PERFORMANCE EVALUATION OF THE TIME–STAMP ORDERING ALGORITHM IN A DISTRIBUTED DATABASE

by

Subir Varma[1]

Electrical Engineering Department and Systems Research Center
University of Maryland, College Park, Maryland 20742

ABSTRACT

Time–stamp ordering is one of the consistency preserving algorithms that is used in distributed data–bases. Baccelli [1] has introduced a queueing model to analyze its performance which incorporates, both the fork–join as well as the resequencing synchronization constraints in its structure. In this paper, we illustrate the power of interpolation approximation technique, by obtaining extremely good approximations for this rather complex model. The heavy traffic approximations are obtained by showing that this model has the same diffusion limit as a system of parallel fork–join queues, the heavy traffic limit for which was obtained in [6]. The light traffic limits are obtained by applying the Reiman–Simon [5] light traffic theory.

[1] The work of this author was supported by a fellowship from the Systems Research Center, University of Maryland

Index Terms: Diffusion approximations, fork–join queues, heavy traffic limits, light traffic limits, resequencing queues, time–stamp ordering.

1. Introduction

Consider a fully replicated distributed database consisting of K storage sites. Assume that the time–stamp ordering consistency preserving algorithm is used in this database. This consists in predefining a total order among the update transactions that operate on a data element and processing them according to this order on all the sites. In order to obtain such a total order, all the access sites are put on a (possibly virtual) token ring, and a token is circulated on this ring. On each access site the update transactions are queued up. When the token with integer value N arrives at a site, it numbers the transactions at that site sequentially with time–stamps from $N + 1$ to $N + M$ (assuming that M transactions were waiting there). The mark of the token is then increased to $N + M$.

Baccelli [1] has introduced a queueing model for this algorithm, which operates as follows (Fig 1). The model is simplified by assuming that the circulation of the token in the access sites is fast enough in comparison to the update transaction arrival process. Once an update transaction receives a time–stamp from a circulating token, it is instantly split into K parts, with each part going to a different storage site. This corresponds to *fork* synchronization constraint. The communication delay between the access site and the storage sites is modeled by an infinite server queue, while the storage sites are modeled by K single server queues. Since the update transactions may arrive at the storage sites in an order different from the one that they were assigned by the circulating token, they may undergo a *resequencing* delay before entering the storage site. The transaction is said to have concluded when each of its K parts have finished their execution at their respective storage sites. This corresponds to a *join* synchronization constraint.

Baccelli [1] has obtained the stability conditions for this queueing model, and also has given techniques which can be used to obtain bounds for the response time. In this paper our aim is to obtain good approximations for the average response time by using techniques from heavy and light traffic approximations. We show that the time–stamp odering model has the same heavy traffic limit as the usual K–dimensional fork–join queue, whose heavy traffic limits were obtained in [6]. We also obtain the light traffic limits for this model by using the Reiman–Simon [5] theory. By combining the heavy traffic limit with the light traffic limit, we are able to obtain good approximations for the entire traffic range.

This paper is organized as follows. The recursions governing the model, as well as its stability conditions are given in Section 2. In Section 3 we obtain the heavy traffic diffusion limit, while in Section 4 we obtain the light traffic limit.

2. Recursive representation of the delays

In this section a recursive representation for the delays in the time–stamp ordering system is provided. These equations were first derived in [1]. For all $n = 0, 1 \ldots$ and $1 \leq j \leq K$, define the following RV's on some probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

u_{n+1} : Inter-arrival time between the $(n+1)^{rst}$ and n^{th} exogenous customers.

v_n^j : Service time requirement of the n^{th} customer to be served in queue j.

d_n^j : Service time requirement of the n^{th} customer to be served in infinite server queue j.

D_n^j : The delay between the n^{th} exogenous customer arrival and the beginning of his service at the j^{th} queue.

W_n^j : Waiting time of the n^{th} exogenous customer in the the resequencing box associated with the infinite server queue j, as well as the buffer of queue j.

T_n : End–to–end delay of the n^{th} exogenous customer.

We assume the system to be initially empty and adopt the convention that the 0^{th} exogenous customer is created at time $t = 0$. The following recursion holds between these variables.

Lemma 2.1. *If the system is initially empty, then for $1 \leq j \leq K$, the recursions*

$$D_0^j = d_0^j$$

$$D_{n+1}^j = \max\{d_{n+1}^j, D_n^j + v_n^j - u_{n+1}\}, \qquad n = 0, 1 \ldots (2.1)$$

$$W_0^j = 0$$

$$W_{n+1}^j = \max\{0, W_n^j + d_n^j - d_{n+1}^j + v_n^j - u_{n+1}\}, \qquad n = 0, 1 \ldots (2.2)$$

hold where the maximum over an emptyset is zero by convention. Moreover the end–to–end delay of the n^{th} customer is given by

$$T_n = \max_{1 \leq j \leq K} \{D_n^j + v_n^j\}. \qquad n = 0, 1 \ldots (2.3)$$

The proof of this lemma may be found in [1].

We make the following assumption.

(I): The sequences $\{u_{n+1}\}_0^\infty$, $\{d_n^j\}_0^\infty$ and $\{v_n^j\}_0^\infty, 1 \le j \le K$, are iid with finite second moments and mutually independent.

For all $n = 0, 1 \ldots$, we set

$$u = \mathbb{E}(u_n) < \infty, \quad \sigma_0^2 = Var(u_n) < \infty$$

$$v^j = \mathbb{E}(v_n^j) < \infty, \quad \sigma_j^2 = Var(v_n^j) < \infty, \quad 1 \le j \le K$$

$$\overline{d}^j = \mathbb{E}(d_n^j) < \infty, \quad \overline{\sigma}_j^2 = Var(d_n^j) < \infty, \quad 1 \le j \le K.$$

We consider the system to be stable if the sequence of delay vectors $\{(D_n^1, \ldots, D_n^K)\}_0^\infty$ converges in distribution as $n \uparrow \infty$ to a proper random vector $(D^1, \ldots, D^K)$.It has been shown in [1], that the condition

$$v^j < u, \quad 1 \le j \le K \tag{2.4}$$

is sufficient to ensure stability of the system.

3. The diffusion limit

In the last section we saw that the the time–stamp ordering system will be stable provided $v^j < u, 1 \le j \le K$. The system is said to be in heavy traffic if $v^j \approx u$ for one or more queues. In this section our objective is to develop heavy traffic diffusion limits for the delay processes in these networks. We shall use the recursions (2.1)–(2.2) to connect the delay processes to partial sums of iid RV's and then use the well known results regarding functional central limit theorems for these partial sums to deduce the corresponding limit theorems for the delay processes by means of the continuous mapping theorem. The main result that we obtain is that the time–stamp ordering system has the same diffusion limit as the usual fork–join queue.

We now consider a sequence of these systems indexed by $r = 1, 2 \ldots$, each of which satisfies condition **(I)**. Moreover assume that:

(II): As $r \uparrow \infty$,

$$\sigma_j(r) \to \sigma_j, \quad 0 \le j \le K$$
$$\overline{\sigma}_j(r) \to \overline{\sigma}_j, \quad 0 \le j \le K$$
$$[u(r) - v^j(r)]\sqrt{r} \to c_j. \quad 1 \le j \le K$$

(III): For some $\epsilon > 0$,

$$\sup_{r,j} \{ \mathbb{E}\{\mid u_1(r) \mid^{2+\epsilon}\}, \mathbb{E}\{\mid v_1^j(r) \mid^{2+\epsilon} \mathbb{E}\{\mid d_1^j(r) \mid^{2+\epsilon}\} < \infty.$$

For $1 \le j \le K$ and $r = 1, 2 \ldots$, define the partial sums

$$V_0^j(r) = 0,$$
$$V_n^j(r) = v_0^j(r) + \ldots + v_{n-1}^j(r), \qquad n = 1, 2 \ldots (3.1a)$$

and

$$U_0(r) = 0,$$
$$U_n(r) = u_1(r) + \ldots + u_n(r). \qquad n = 1, 2 \ldots (3.1b)$$

For $r = 1, 2 \ldots$, define the stochastic processes $\xi^j(r) \equiv \{\xi_t^j(r), t \ge 0\}, 0 \le j \le K$, with sample paths in $D[0, \infty)$ by

$$\xi_t^0(r) = \frac{U_{[rt]}(r) - u(r)[rt]}{\sqrt{r}}, \quad t \ge 0 \qquad (3.2a)$$

$$\xi_t^j(r) = \frac{V_{[rt]}^j(r) - v^j(r)[rt]}{\sqrt{r}}, \quad 1 \le j \le K, \quad t \ge 0. \qquad (3.2b)$$

Let $\xi^j \equiv \{\xi_t^j, t \ge 0\}, 0 \le j \le K$, be $K+1$ independent Wiener processes. Lemma 3.1 shows that the random functions defined in (3.2), converge weakly to these Wiener processes.

Lemma 3.1. *As* $r \uparrow \infty$,

$$(\xi^0(r), \xi^1(r), \ldots, \xi^K(r)) \Rightarrow (\sigma_0\xi^0, \sigma_1\xi^1, \ldots, \sigma_B\xi^K) \qquad (3.3)$$

in $D[0, \infty)^{K+1}$.

Proof. Equation (3.3) follows directly by Prohorov's functional central limit theorem for triangular arrays [4] under assumptions **(I)**–**(III)**.

∎

For $r = 1, 2 \ldots$, we set

$$S_0^j(r) = 0$$
$$S_n^j(r) = V_n^j(r) - U_n(r), \qquad n = 1, 2 \ldots (3.4)$$

and define the following stochastic processes $\{\zeta^j(r) \equiv \{\zeta_t^j(r), t \geq 0\}, 1 \leq j \leq K$,with sample paths on $D[0, \infty)$ by

$$\zeta_t^j(r) = \frac{S_{[rt]}^j(r)}{\sqrt{r}}, \quad 1 \leq j \leq K, \quad t \geq 0. \tag{3.5}$$

We also define the stochastic processes $\zeta^j \equiv \{\zeta^j, t \geq 0\}, 1 \leq j \leq K$, by

$$\zeta_t^j = \sigma_j \xi_t^j - \sigma_0 \xi_t^0 - c_j t, \quad 1 \leq j \leq K, \quad t \geq 0. \tag{3.6}$$

The process $(\zeta^1, \ldots, \zeta^K)$ is a K–dimensional diffusion process with drift vector c and covariance matrix R given by

$$c = (-c_1, \ldots, -c_K) \tag{3.7}$$

and

$$R = \begin{pmatrix} \sigma_1^2 + \sigma_0^2 & \sigma_0^2 & \ldots & \sigma_0^2 \\ \sigma_0^2 & \sigma_2^2 + \sigma_0^2 & \ldots & \sigma_0^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_0^2 & \sigma_0^2 & \ldots & \sigma_K^2 + \sigma_0^2 \end{pmatrix}. \tag{3.8}$$

The next result shows that the stochastic processes $(\zeta^1(r), \ldots, \zeta^K(r))$ converge weakly to $(\zeta^1, \ldots, \zeta^K)$.

Lemma 3.2. *As* $r \uparrow \infty$,

$$(\zeta^1(r), \ldots, \zeta^K(r)) \Rightarrow (\zeta^1, \ldots, \zeta^K) \tag{3.9}$$

in $D[0, \infty)^K$.

Proof. Fix $r \geq 1$ and $t \geq 0$. For all $1 \leq k \leq K$, we see from (3.4)–(3.5) that

$$\begin{aligned}\zeta_t^k(r) &= \frac{V_{[rt]}^k(r) - U_{[rt]}}{\sqrt{r}} \\ &= \frac{V_{[rt]}^k(r) - v^k(r)[rt]}{\sqrt{r}} - \frac{U_{[rt]}^k(r) - u(r)[rt]}{\sqrt{r}} - \frac{[rt][u(r) - v^k(r)]}{\sqrt{r}} \\ &= \xi_t^k(r) - \xi_t^0(r) - \frac{[rt]}{r}[u(r) - v^k(r)]\sqrt{r}\end{aligned}$$

From assumption **(II)** it is clear that as $r \uparrow \infty$,

$$\frac{[rt]}{r}[u(r) - v^k(r)]\sqrt{r} \to c_k t, \quad 1 \leq k \leq K$$

and we conclude to (3.9) by invoking Lemma 3.1 and the continuous mapping theorem [2].

∎

For $r = 1, 2 \ldots$, we define the stochastic processes $\mu^j(r) \equiv \{\mu_t^j(r), t \geq 0\}, 1 \leq j \leq K$, with sample paths in $D[0, \infty)$, by setting

$$\mu_t^j(r) = \frac{W_{[rt]}^j(r)}{\sqrt{r}}, \quad 1 \leq j \leq K, \quad t \geq 0. \tag{3.10}$$

The processes $\mu^j \equiv \{\mu_t^j, t \geq 0\}, 1 \leq j \leq K$, are now defined by

$$\mu^j = g(\zeta^j), \quad 1 \leq j \leq K \tag{3.11}$$

We now present the main result of this section.

Theorem 3.1. *As $r \uparrow \infty$,*

$$(\mu^1(r), \ldots, \mu^K(r)) \Rightarrow (\mu^1, \ldots, \mu^K) \tag{3.12}$$

in $D[0, \infty)^K$.

Before providing a proof for Theorem 3.1, we present the following two corollaries. For $r = 1, 2 \ldots$, we define the stochastic processes $\eta^j(r) \equiv \{\eta_t^j(r), t \geq 0\}$ with sample paths in $D[0, \infty)$, by

$$\eta_t^j(r) = \frac{D_{[rt]}^j(r)}{\sqrt{r}}, \quad 1 \leq j \leq K, \quad t \geq 0 \tag{3.13}$$

Corollary 3.1. *As* $r \uparrow \infty$,

$$(\eta^1(r), \ldots, \eta^K(r)) \Rightarrow (\mu^1, \ldots, \mu^K) \tag{3.14}$$

in $D[0,\infty)^K$.

Proof. Note that for all $r = 1, 2 \ldots$,

$$D_n^j(r) = W_n^j(r) + d_n^j(r), \quad 1 \leq j \leq K \qquad\qquad n = 0, 1 \ldots$$

so that for all $r = 1, 2 \ldots$,

$$\eta_t^j(r) = \mu_t^j(r) + \frac{d_{[rt]}^j(r)}{\sqrt{r}}, \quad 1 \leq j \leq K, \quad t \geq 0 \tag{3.15}$$

We obtain (3.14) from (3.12) and (3.15) by applying the continuous mapping theorem and the converging together theorem [2]. ∎

For $r = 1, 2 \ldots$, we introduce the stochastic processes $\kappa(r) \equiv \{\kappa_t(r), t \geq 0\}$ with sample paths in $D[0,\infty)$ by

$$\kappa_t(r) = \frac{T_{[rt]}(r)}{\sqrt{r}}, \quad t \geq 0. \tag{3.16}$$

Corollary 3.2. *As* $r \uparrow \infty$,

$$\kappa(r) \Rightarrow \max_{1 \leq j \leq K} \eta^i \tag{3.17}$$

in $D[0,\infty)$.

Proof. Using the fact that for all $r = 1, 2 \ldots$

$$\kappa_t(r) = \max_{1 \leq j \leq K} \{\eta_t^i(r) + \frac{v_{[rt]}^i}{\sqrt{r}}\}, \quad t \geq 0 \tag{3.18}$$

we obtain (3.18) from (3.14) by applying the continuous mapping theorem and the converging together theorem. ∎

We now proceed with the proof for Theorem 3.1.

Proof. Fix $r = 1, 2 \ldots$. For $1 \leq j \leq K$, we can write the recursion (2.2) for the waiting time sequence as

$$W_0^j(r) = 0,$$
$$W_{n+1}^j(r) = \max\{0, W_n^j(r) + X_{n+1}^j(r)\}, \qquad n = 0, 1, \ldots \quad (3.19)$$

where

$$X_{n+1}^j(r) = d_n^j(r) - d_{n+1}^j(r) + v_n^j(r) - u_{n+1}(r). \qquad n = 0, 1 \ldots \quad (3.20)$$

By successive substitutions, we obtain

$$W_n^j(r) = \max\{0, X_n^j(r), X_n^j(r) + X_{n-1}^j(r), \ldots, X_n^j(r) + \ldots + X_1^j(r)\}. \quad n = 0, 1 \ldots \quad (3.21)$$

Let

$$Z_0^j(r) = 0,$$
$$Z_n^j(r) = \sum_{i=1}^{n} X_i^j(r). \qquad n = 1, 2 \ldots \quad (3.22)$$

It follows that

$$W_n^j(r) = Z_n^j(r) - \min_{0 \leq k \leq n} Z_k^j(r). \qquad n = 0, 1 \ldots \quad (3.23)$$

Note that

$$Z_n^j(r) = D_0^j(r) - D_n^j(r) + S_n^j(r). \qquad n = 0, 1 \ldots \quad (3.24)$$

For $r = 1, 2 \ldots$ we introduce the stochastic process $\rho^j(r) \equiv \{\rho_t^j(r), t \geq 0\}, 1 \leq j \leq K$, with sample paths in $D[0, \infty)$ by

$$\rho_t^j(r) = \frac{Z_{[rt]}^j(r)}{\sqrt{r}}, \quad 1 \leq j \leq K, \quad t \geq 0. \quad (3.25)$$

From (3.23) and (3.25) it follows that

$$\mu_t^j(r) = g(\rho^j(r))_t, \quad t \geq 0. \quad (3.26)$$

Hence by the continuous mapping theorem, in order to prove (3.13), it is sufficient to show that

$$(\rho^1(r), \ldots, \rho^K(r)) \Rightarrow (\zeta^1, \ldots, \zeta^K) \quad (3.27)$$

in $D[0,\infty)^K$, as $r \uparrow \infty$. But this follows by (3.9), (3.21) and the converging together theorem. ∎

Corollary 3.2 implies that the end–to–end delay sequence in the time–stamp ordering queue has the same heavy traffic diffusion limit as the end–to–end delay sequence in a fork–join queue [6]. The following formula was given in [6] for the heavy traffic limit for the end–to–end delay of a homogeneous fork–join queue. Assume that the service processes have a distribution with rate μ and variance σ^2, while the arrival process has rate λ and limiting variance (as $\lambda \uparrow \mu$) σ_0^2. Let

$$\beta = \frac{\sigma_0^2}{\sigma^2 + \sigma_0^2}. \tag{3.28}$$

Denoting the average end–to–end delay when the arrival rate is λ by $\overline{T}(\lambda)$, then

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda) \overline{T}_K(\lambda)$$

$$= \left[H_K + (4V_K - 3H_K - 1)\beta + 2(1 + H_K - 2V_K)\beta^2\right] \frac{\sigma^2 + \sigma_0^2}{2} \mu^2, \quad 0 \le \beta \le 1. \qquad K = 2, 3 \ldots (3.29)$$

where

$$H_K = \sum_{k=1}^{K} \frac{1}{k} \tag{3.30}$$

and

$$V_K = \sum_{r=1}^{K} \binom{K}{r} (-1)^{r-1} \sum_{m=1}^{r} \binom{r}{m} \frac{(m-1)!}{r^{m+1}}. \qquad K = 2, 3 \ldots (3.31)$$

4. The interpolation approximation

In this section we consider the special case of a homogeneous time–stamp ordering system with Poisson arrivals and exponential service and disordering distributions. We obtain the light traffic limits for the end–to–end delay of this model and combine it with the heavy traffic limit of the previous section, to obtain good approximations for the entire traffic range.

We shall assume that the batches arrive into the system according to a Poisson process with rate λ, the K infinite server queues have service times with exponential rate ν, and the K single server queues also have service times with exponential rate μ. Specializing (3.29) to this case, we obtain

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda) \overline{T}_K(\lambda) = V_K. \qquad K = 2, 3 \ldots (4.1)$$

Before the light traffic limits can be obtained, we have to prove that the system is admissible in the sense of Reiman and Simon [5]. This was done in [6], to which the reader is referred to for further details.

We first calculate $\overline{T}_K(0)$. Consider the batch arriving at $t = 0$. Let $d_1, \ldots, d_K$ be its disordering delays and $s_1, \ldots, s_K$ be its service times at the K queues. Since this batch does not experience interference from any other customer, i.e., it does not experience any queueing or resequencing delay, it follows that

$$\overline{T}_K(0) = \mathbb{E} \max_{1 \le k \le K} (d_k + s_k). \qquad (4.2)$$

Each of $d_k + s_k, 1 \le k \le K$, has a common distribution $F(z)$, given by

$$F(z) = 1 + \frac{\mu}{\nu - \mu} \exp(-\nu z) - \frac{\nu}{\nu - \mu} \exp(-\mu z), \quad z \ge 0 \qquad (4.3)$$

so that

$$\begin{aligned}
\overline{T}_K(0) &= \int_0^\infty \left[1 - (1 + \frac{\mu}{\nu - \mu} e^{-\nu z} - \frac{\nu}{\nu - \mu} e^{-\mu z})^K \right] dz \\
&= \int_0^\infty \left[1 - \sum_{r=0}^{K} \binom{K}{r} [\frac{\mu}{\nu - \mu} e^{-\nu z} - \frac{\nu}{\nu - \mu} e^{-\mu z}]^r \right] dz \\
&= \sum_{r=1}^{K} \binom{K}{r} \sum_{m=0}^{r} \binom{r}{m} (-1)^{m+1} (\frac{\mu}{\nu - \mu})^{r-m} (\frac{\nu}{\nu - \mu})^m \\
&\quad \times \int_0^\infty e^{-(m\mu + (r-m)\nu) z} dz \\
&= \sum_{r=1}^{K} \binom{K}{r} \sum_{m=0}^{r} \binom{r}{m} (-1)^{m+1} (\frac{\mu}{\nu - \mu})^{r-m} (\frac{\nu}{\nu - \mu})^m \frac{1}{m\mu + (r-m)\nu}.
\end{aligned}$$

Let $L_K(\mu, \nu)$ denote the right hand side of this last equation, so that

$$\overline{T}_K(0) = L_K(\mu, \nu). \tag{4.6}$$

Values of $L_K(1,2), 1 \le K \le 10$ are given in Table II.

We now proceed to calculate $\overline{T}'_K(0)$. Let

$$T(t, d_1, \ldots, d_K, s_1, \ldots, s_K, \overline{d}_1, \ldots, \overline{d}_K, \overline{s}_1, \ldots, \overline{s}_K)$$

be the response time of the batch that arrives at time $t = 0$ with service times $s_1, \ldots, s_K$ and disordering delays $d_1, \ldots, d_K$ given that another customer arrives at time t with disordering delays $\overline{d}_1, \ldots, \overline{d}_K$ and service time $\overline{s}_1, \ldots, \overline{s}_K$. Then it is not difficult to see that

$$T(t, d_1, \ldots, d_K, s_1, \ldots, s_K, \overline{d}_1, \ldots, \overline{d}_K, \overline{s}_1, \ldots, \overline{s}_K)$$
$$= \begin{cases} \max_{1 \le k \le K}(d_k + s_k), & \text{if } t > 0 \\ \max_{1 \le k \le K}[\max(d_k, t + \overline{d}_k + \overline{s}_k) + s_k] & \text{if } t \le 0. \end{cases} \tag{4.5}$$

Define the RV's $X_k, 1 \le k \le K$, as

$$X_k = t + \overline{d}_k + \overline{s}_k. \tag{4.6}$$

Then it can be shown that each $X_k, 1 \le k \le K$, has the following distribution,

$$F_X(x) = 1 - \frac{\nu}{\nu - \mu} \exp(\mu(t - x)) + \frac{\mu}{\nu - \mu} \exp(\nu(t - x)), \quad x \ge t. \tag{4.7}$$

Next define the RV's $Y_k, 1 \le k \le K$ as

$$Y_k = \max(d_k, X_k) \tag{4.8}$$

Since the RVs d_k and X_k are independent for $1 \le k \le K$, each $Y_k, 1 \le k \le K$, has the distribution F_Y given by

$$\begin{aligned} F_Y(x) =& [1 - \frac{\nu}{\nu - \mu} e^{\mu(t-x)} + \frac{\mu}{\nu - \mu} e^{\nu(t-x)}](1 - e^{-\nu x}) \\ =& 1 - \frac{\nu}{\nu - \mu} e^{\mu(t-x)} + \frac{\mu}{\nu - \mu} e^{\nu(t-x)} - e^{-\nu x} \\ &+ \frac{\nu}{\nu - \mu} e^{\mu t} e^{-(\nu+\mu)x} - \frac{\mu}{\nu - \mu} e^{\nu t} e^{-2\nu x}, \quad x \ge 0. \end{aligned} \tag{4.9}$$

Lastly define the RV's $R_k, 1 \le k \le K$, as

$$R_k = Y_k + s_k. \tag{4.10}$$

Taking into account that the RV's s_k and Y_k are independent for $1 \le k \le K$, it can be shown that each $R_k, 1 \le k \le K$,

$$\begin{aligned} F_R(x) =& 1 + \frac{\mu}{\nu-\mu}e^{-\nu x} - \frac{\nu}{\nu-\mu}e^{-\mu x} - \frac{\nu\mu}{\nu-\mu}e^{\mu t}xe^{-\mu x} \\ &+ \frac{\mu^2(3\nu-2\mu)}{(\nu-\mu)^2(2\nu-\mu)}e^{\nu t}e^{-\mu x} - \frac{\mu^2}{(\nu-\mu)^2}e^{\nu t}e^{-\nu x} \\ &+ \frac{\mu}{\nu-\mu}e^{\mu t}e^{-\mu x} - \frac{\mu}{\nu-\mu}e^{\mu t}e^{-(\nu+\mu)x} \\ &- \frac{\mu^2}{(2\nu-\mu)(\nu-\mu)}e^{\nu t}e^{-2\nu x}, \quad x \ge 0. \end{aligned} \tag{4.11}$$

Note that from (4.5), (4.6), (4.8) and (4.10),

$$T = \max_{1 \le k \le K} R_k \quad \text{if} \quad t \le 0 \tag{4.12}$$

where the left hand side of (4.5) has been abbreviated to T. Since the RV's $R_k, 1 \le k \le K$, are independent, we obtain that

$$\mathbb{P}(T \le x) = \prod_{k=1}^{K} \mathbb{P}(R_k \le x) = F_R^K(x), \quad x \ge 0. \tag{4.13}$$

Using the fact [5], that

$$\overline{T}'_K(0) = \int_{t=-\infty}^{\infty}\int_{x=0}^{\infty}(1 - \mathbb{P}(T \le x))dxdt$$

it can be shown after some calculations that

$$\overline{T}'_K(0) = -\int_{t=-\infty}^{0}\int_{x=0}^{\infty}\sum_{r=1}^{K}\binom{K}{r}U^{K-r}V^r dxdt \tag{4.14}$$

where

$$U = 1 + \frac{\mu}{\nu - \mu}\exp(-\nu x) - \frac{\nu}{\nu - \mu}\exp(-\mu x) \tag{4.15}$$

and

$$\begin{aligned} V =& \frac{\mu^2(3\nu - 2\mu)}{(\nu-\mu)^2(2\nu-\mu)}e^{\nu t}e^{-\mu x} - \frac{\nu\mu}{\nu-\mu}e^{\mu t}xe^{-\mu x} \\ &- \frac{\mu^2}{(\nu-\mu)^2}e^{\nu t}e^{-\nu x} + \frac{\mu}{\nu-\mu}e^{\mu t}e^{-\mu x} \\ &- \frac{\mu}{\nu-\mu}e^{\mu t}e^{-(\nu+\mu)x} - \frac{\mu^2}{(2\nu-\mu)(\nu-\mu)}e^{\nu t}e^{-2\nu x}. \end{aligned} \tag{4.16}$$

It can be shown that

$$\begin{aligned} U^{K-r} =& \sum_{m_1=0}^{K-r}\binom{K-r}{m_1}\sum_{m_2=0}^{m_1}\binom{m_1}{m_2}(-1)^{m_2}(\frac{\mu}{\nu-\mu})^{m_1-m_2}(\frac{\nu}{\nu-\mu})^{m_2} \\ &\times e^{-(\nu(m_1-m_2)+\mu m_2)x} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} V^r =& \sum_{k_1=0}^{r}(-1)^{k_1}\binom{r}{k_1}\sum_{k_2=0}^{r-k_1}\binom{r-k_1}{k_2}(\frac{\mu}{\nu-\mu})^{r-k_1-k_2} \\ &\times [\frac{\mu^2(3\nu-2\mu)}{(\nu-\mu)^2(2\nu-\mu)}]^{k_2}e^{(\mu(r-k_1-k_2)+\nu k_2)t}e^{-\mu x(r-k_1)} \\ &\times \sum_{k_3=0}^{k_1}\binom{k_1}{k_3}\sum_{k_4=0}^{k_1-k_3}\binom{k_1-k_3}{k_4}(\frac{\nu\mu}{\nu-\mu})^{k_4}[\frac{\mu^2}{(\nu-\mu)^2}]^{k_1-k_3-k_4} \\ &\times e^{(\mu k_4+\nu(k_1-k_3-k_4)t}x^{k_4}e^{-(\mu k_4+\nu(k_1-k_3-k_4))x} \\ &\times \sum_{k_5=0}^{k_3}\binom{k_3}{k_5}(\frac{\mu}{\nu-\mu})^{k_5}[\frac{\mu^2}{(\nu-\mu)(2\nu-\mu)}]^{k_3-k_5} \\ &\times e^{(\mu k_5+\nu(k_3-k_5))t}e^{-(\mu k_5+\nu(2k_3-k_5))x}. \end{aligned} \tag{4.18}$$

From (4.14), (4.17) and (4.18) it follows that

$\overline{T}'_K(0)$

$$= -\sum_{r=1}^{K}\binom{K}{r}\sum_{m_1=0}^{K-r}\binom{K-r}{m_1}\sum_{m_2=0}^{m_1}\binom{m_1}{m_2}(-1)^{m_2}(\frac{\mu}{\nu-\mu})^{m_1-m_2}(\frac{\nu}{\nu-\mu})^{m_2}$$

$$\times\sum_{k_1=0}^{r}(-1)^{k_1}\binom{r}{k_1}\sum_{k_2=0}^{r-k_1}\binom{r-k_1}{k_2}(\frac{\mu}{\nu-\mu})^{r-k_1-k_2}[\frac{\mu^2(3\nu-2\mu)}{(\nu-\mu)^2(2\nu-\mu)}]^{k_2}$$

$$\times\sum_{k_3=0}^{k_1}\binom{k_1}{k_3}\sum_{k_4=0}^{k_1-k_3}\binom{k_1-k_3}{k_4}(\frac{\nu\mu}{\nu-\mu})^{k_4}[\frac{\mu^2}{(\nu-\mu)^2}]^{k_1-k_3-k_4}$$

$$\times\sum_{k_5=0}^{k_3}\binom{k_3}{k_5}(\frac{\mu}{\nu-\mu})^{k_5}[\frac{\mu^2}{(\nu-\mu)(2\nu-\mu)}]^{k_3-k_5}$$

$$\times\int_0^{\infty}x^{k_4}e^{-(\mu(r+m_2-k_1+k_4+k_5)+\nu(m_1-m_2+k_1+k_3-k_4-k_5))x}dx$$

$$\times\int_{-\infty}^{0}e^{(\mu(r-k_1-k_2+k_4+k_5)+\nu(k_1+k_2-k_4-k_5))t}dt$$

$$= -\sum_{r=1}^{K}\binom{K}{r}\sum_{m_1=0}^{K-r}\binom{K-r}{m_1}\sum_{m_2=0}^{m_1}\binom{m_1}{m_2}(-1)^{m_2}(\frac{\mu}{\nu-\mu})^{m_1-m_2}(\frac{\nu}{\nu-\mu})^{m_2}$$

$$\times\sum_{k_1=0}^{r}(-1)^{k_1}\binom{r}{k_1}\sum_{k_2=0}^{r-k_1}\binom{r-k_1}{k_2}(\frac{\mu}{\nu-\mu})^{r-k_1-k_2}[\frac{\mu^2(3\nu-2\mu)}{(\nu-\mu)^2(2\nu-\mu)}]^{k_2}$$

$$\times\sum_{k_3=0}^{k_1}\binom{k_1}{k_3}\sum_{k_4=0}^{k_1-k_3}\binom{k_1-k_3}{k_4}(\frac{\nu\mu}{\nu-\mu})^{k_4}[\frac{\mu^2}{(\nu-\mu)^2}]^{k_1-k_3-k_4}$$

$$\times\sum_{k_5=0}^{k_3}\binom{k_3}{k_5}(\frac{\mu}{\nu-\mu})^{k_5}[\frac{\mu^2}{(\nu-\mu)(2\nu-\mu)}]^{k_3-k_5}$$

$$\times\frac{k_4!}{[\mu(r+m_2-k_1+k_4+k_5)+\nu(m_1-m_2+k_1+k_3-k_4-k_5)]^{k_4+1}}$$

$$\times\frac{1}{\mu(r-k_1-k_2+k_4+k_5)+\nu(k_1+k_2-k_4-k_5)}. \tag{4.21}$$

We shall denote the right hand side of (4.21) as $G_K(\mu,\nu)$ so that

$$\overline{T}'_K(0) = G_K(\mu,\nu). \tag{4.22}$$

Values of $G_K(1,2), 2 \le K \le 10$ are given in Table II.

Finally combining (4.1), (4.4) and (4.20), we obtain the following first order approximation to the average response time of the time stamp ordering model,

$$\hat{T}_K(\lambda) = \frac{\mu L_K(\mu,\nu)}{\mu-\lambda} + [\mu G_K(\mu,\nu) - L_K(\mu,\nu)]\,\frac{\lambda}{\mu-\lambda} + \left[V_K - \mu^2 G_K(\mu,\nu)\right](\frac{\lambda}{\mu})^2 \frac{1}{\mu-\lambda}, \quad 0 \le \lambda < \mu. \tag{4.23}$$

This approximation agrees extremely well with simulation results (see Table I).

Acknowledgment: This work was done as part of the author's Ph.D. dissertation. He would like to thank his advisor Prof. Armand Makowski for his help and support and his continued interest in this work.

References

[1] F. Baccelli,"A queueing model for timestamp ordering in a distributed system," *Performance '87*, Brussels, pp. 413-431 (1987).

[2] P. Billingsley, *"Convergence of probability measures,"* J. Wiley and Sons, New York, (1968).

[3] E. Kyprianou, "The virtual waiting time of the $GI/G/1$ queue in heavy traffic," *Adv. Appl. Prob.*, No. 3, pp. 249-268 (1971).

[4] Y. Prohorov,"Convergence of random processes and limit theorems of probability theory," *Theor. Probability Appl.*, Vol. 1, pp. 157-214 (1956).

[5] M.I. Reiman and B. Simon, "Open queueing systems in light traffic," *Maths. of Oper. Res.*, Vol. 14, No. 1, pp. 26-59 (1989).

[6] S. Varma, "Heavy and light traffic approximations for queues with synchronization constraints," PhD Thesis, University of Maryland, (1990).

Table I

In these tables, approximation (4.23) is compared with simulation results for the case $\mu = 1, \nu = 2$ and $K = 2, 3, 5$ and 10.

λ	$\overline{T}_2(\lambda)$	$\hat{T}_2(\lambda)$	% Error
0.1	2.25 ± 0.008	2.23	0.89
0.2	2.46 ± 0.012	2.43	1.22
0.3	2.73 ± 0.016	2.67	2.19
0.4	3.08 ± 0.017	3.00	2.59
0.5	3.56 ± 0.036	3.46	2.80
0.6	4.27 ± 0.062	4.15	2.81
0.7	5.40 ± 0.112	5.30	1.88
0.8	7.59 ± 0.24	7.59	0.00
0.9	14.54 ± 0.31	14.47	0.48

λ	$\overline{T}_3(\lambda)$	$\hat{T}_3(\lambda)$	% Error
0.1	2.64 ± 0.007	2.63	0.38
0.2	2.88 ± 0.011	2.85	1.04
0.3	3.17 ± 0.015	3.14	0.95
0.4	3.57 ± 0.022	3.52	1.40
0.5	4.11 ± 0.036	3.89	5.35
0.6	4.92 ± 0.059	4.86	1.22
0.7	6.31 ± 0.111	6.19	1.90
0.8	9.13 ± 0.295	8.85	3.07
0.9	17.63 ± 1.23	16.82	4.59

λ	$\overline{T}_5(\lambda)$	$\hat{T}_5(\lambda)$	% Error
0.1	2.64 ± 0.007	2.63	0.38
0.1	3.14 ± 0.009	3.14	0.06
0.2	3.41 ± 0.012	3.40	0.32
0.3	3.76 ± 0.017	3.73	0.80
0.4	4.22 ± 0.026	4.18	0.99
0.5	4.87 ± 0.041	4.88	0.20
0.6	5.83 ± 0.068	5.73	1.71
0.7	7.39 ± 0.12	7.28	1.48
0.8	10.44 ± 0.28	10.39	0.57
0.9	19.96 ± 0.39	19.69	1.35

λ	$\overline{T}_{10}(\lambda)$	$\hat{T}_{10}(\lambda)$	% Error
0.1	3.84 ± 0.008	3.84	0.08
0.2	4.16 ± 0.011	4.14	0.48
0.3	4.57 ± 0.016	4.53	0.87
0.4	5.11 ± 0.019	5.05	1.17
0.5	5.86 ± 0.026	5.78	1.36
0.6	6.99 ± 0.076	6.88	1.57
0.7	8.89 ± 0.15	8.70	2.13
0.8	12.73 ± 0.37	12.34	3.06
0.9	24.25 ± 0.43	23.26	4.08

Table II

K	$L_K(1,2)$	$G_K(1,2)$
2	2.08	1.39
3	2.45	1.62
4	2.72	1.77
5	2.93	1.88
6	3.10	1.97
7	3.25	2.04
8	3.38	2.10
9	3.49	2.15
10	3.60	2.19

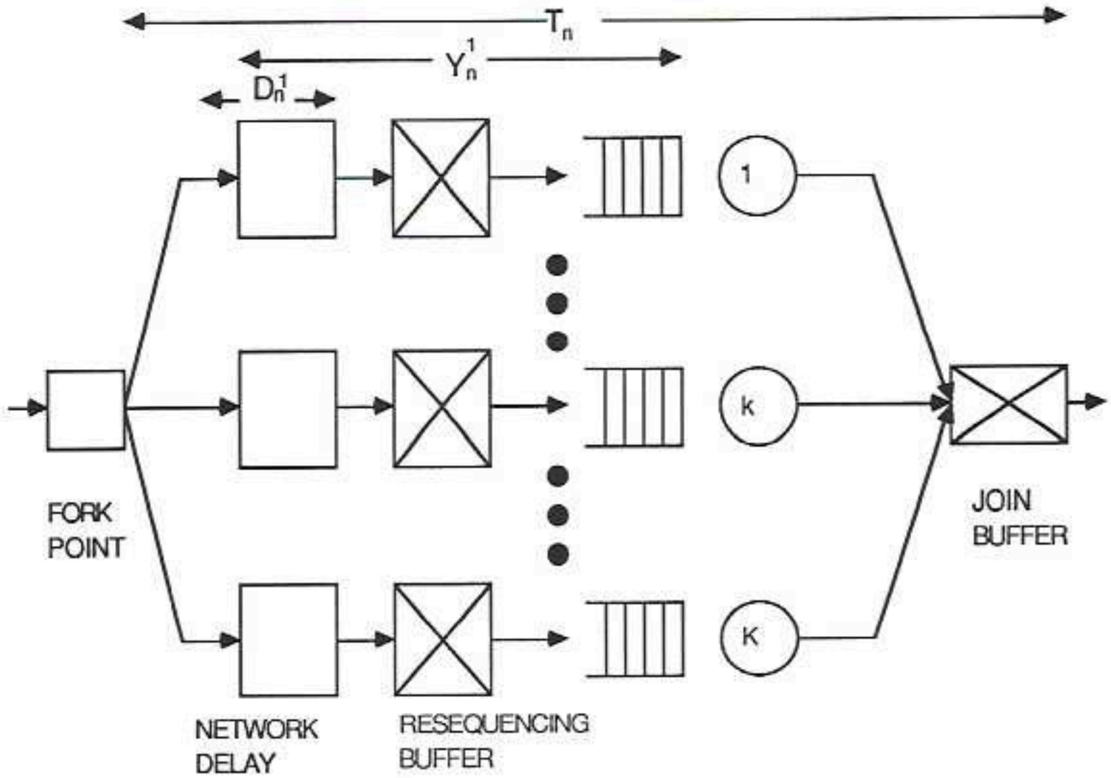

Fig 1. The time-stamp ordering system