To Appear in Performance Evaluation, May 1991

A MATRIX GEOMETRIC SOLUTION TO A RESEQUENCING PROBLEM

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ABSTRACT

Consider a $M/M/2/B$ queue with heterogenous servers which operates under the resequencing constraint that customers should leave the system in the order in which they entered it. A matrix geometric solution to the steady state buffer occupation probabilities of this system is provided by noting that the infinitesmal generator matrix possesses a block diagonal structure.

Keywords: Resequencing, Matrix geometric solution.

* The work of this author was supported partially through NSF Grants NSFD CDR-85-00108 and ECS-83-

1. Introduction

Consider a $M/M/2/B$ queue with heterogeneous servers which operates under the resequencing constraint that the customers should leave the system in the order in which they entered it. If a customer goes out of sequence after receiving service, it waits in a special buffer, the so-called resequencing buffer, until all customers who entered the queue prior to it have completed service, at which time it leaves the system. The queueing system described above can be used to model the communication between two nodes in a computer network, which are connected together by two independent channels. Multiple channels are useful because if one of the links becomes faulty, then the other link can take on its function, thus paving the way for better fault tolerance. Multiple paths also help in distributing the traffic more evenly in the network. Networks such as IBM's Systems Network Architecture (SNA) employ multiple paths between nodes. However our model does not account for the fact that there are multiple virtual circuits that use the same two links, nor does it account for hop–level flow control protocols between the nodes.

One of the first models to incorporate resequencing into the two server queue was that of Lien [4]. By using a clever extension of the state space for the usual $M/M/2$ queue, he obtained a closed form expression for the average resequencing delay. Later Iliadis and Lien [5], [6] obtained expressions for the distribution of the resequencing delay for the case when the customers are allocated to the servers according to a threshold type of policy. Baccelli, Gelenbe and Plateau [1], Harrus and Plateau [3], Kamoun, Kleinrock and Muntz [7] and Varma [12] have analyzed resequencing systems in which the disordering is due to infinite server queues. Gun and Jean Marie [2], Yum and Ngai [13] and Varma [11], [12] have analyzed resequencing systems in which the disordering is due to finite server queues.

In all the work mentioned above, the emphasis was on obtaining the distribution of the resequencing delay. However, the problem of obtaining the resequencing buffer occupation probability distribution is also important from the practical point of view, since it would help the designer in choosing the size of the resequencing buffer in a appropriate way to minimize overflow. In this paper we take a step in this direction by obtaining an expression for this distribution for the case when the disordering is due to a $M/M/2/B$ queue. We do so by noting that the infinitesmal generator matrix for the system has a block diagonal structure, which yields a matrix-geometric solution for the steady state probabilities.

The rest of the paper is organized as follows. In Section 2 we give a Markovian state space description of the model. In Section 3 we present the corresponding equations for the steady-state probabilities. In Section 4 we give an exact solution to these equations for the special case of $B = 0$. In Section 5, by using matrix-geometric techniques, we solve them for an arbitrary yet finite value of B. Some numerical results are presented in Section 6.

2. A Markovian State Space Description

Consider a $M/M/2/B$ queue with arrival rate λ, and service rates of magnitude μ_1 and μ_2 for servers one and two, respectively. Assume that $\mu_1 \geq \mu_2$ so that server one (1) and server two (2) can be called the fast and slow servers, respectively. Pose

$n =$ number of customers in the main queue buffer.

$e_1 = 1$ (resp. 0) if the faster server is busy (resp. idle).

$e_2 = 1$ (resp. 0) if the slower server is busy (resp. idle).

$m =$ number of customers in the resequencing buffer.

The variables (n, e_1, e_2, m) do not constitute a Markovian description of the system, since there is no way to take into account the effect on m by a service completion at either server. Due to the synchronization constraint on the output customer stream, we need a state variable which captures this effect. A clever way of defining this state which was first given by Luke Lien [1], is now presented. The additional information needed to get a Markovian state space description is the specification of which of the two customers presently in service, started receiving service earlier. This is exactly what the fifth state variable, denoted by Z, specifies with

$Z = I$ if the fast server (1) is serving the customer which entered the system earlier. We shall refer to this as being an *in-sequence* state.

$Z = O$ if the slow server (2) is serving the customer which entered the system earlier. We shall refer to this as being an *out-of-sequence* state.

When there is a single customer in the system, we shall adopt the same notation with the interpretation that $Z = I$ if the customer is with the fast server and $Z = O$ if the customer is with the slow server.

The reader will readily check that (n, e_1, e_2, m, Z) provides a complete Markovian state space description of the system. The state variables (n, e_1, e_2, m, Z) belong to the space

$$E = \{0\} \cup I\!N \times \{0,1\} \times \{0,1\} \times I\!N \times \{I, O\}$$

where $\{0\}$ is the 'empty' state.

If the system is in a in-sequence state $(Z = I)$, then a departure from server 2 leads to an increase in the number of customers in the resequencing box by one ($m \to m+1$), since the customer who arrived earlier is being served by server 1. On the other hand, a departure from server 1 empties all the customers in the resequencing buffer ($m \to 0$), and changes the state to an out-of-sequence state (if there is a customer in service in server 2). By a similar reasoning, if the system is in an out-of-sequence state ($Z = O$), a departure from server 1 leads to an increase in the number of customers in the resequencing box ($m \to m+1$), while a departure from server 2 empties the resequencing box ($m \to 0$).

3. The State Space Equations

In this section we proceed to write down the equations for the steady state probabilities for the Markov Chain associated with the $M/M/2/B$ queue with resequencing.

1. The equilibrium equation at the origin.

$$\lambda P(0) = \mu_1 \sum_{j=0}^{\infty} P(0,1,0,j,I) + \mu_2 \sum_{j=0}^{\infty} P(0,0,1,j,O) \tag{3.1}$$

2. The equilibrium equations for the states for which $Z = I$.

(a) For $0 < i < B, j > 0, e_1 = 1, e_2 = 1$.

$$(\lambda + \mu_1 + \mu_2)P(i,1,1,j,I) = \mu_2 P(i+1,1,1,j-1,I) + \lambda P(i-1,1,1,j,I) \tag{3.2a}$$

(b) For $i = B, j \geq 0, e_1 = 1, e_2 = 1$.

$$(\mu_1 + \mu_2)P(B,1,1,j,I) = \lambda P(B-1,1,1,j,I) \tag{3.2b}$$

(c) For $i = 0, j > 0, e_1 = 1, e_2 = 1$.

$$(\lambda + \mu_1 + \mu_2)P(0,1,1,j,I) = \mu_2 P(1,1,1,j-1,I) + \lambda P(0,1,0,j,I) \tag{3.2c}$$

(d) For $i = 0, j > 0, e_1 = 1, e_2 = 0$.

$$(\lambda + \mu_1)P(0,1,0,j,I) = \mu_2 P(0,1,1,j-1,I) \tag{3.2d}$$

(e) For $0 < i < B, j = 0, e_1 = 1, e_2 = 1$.

$$(\lambda + \mu_1 + \mu_2)P(i,1,1,0,I) = \mu_2 \sum_{j=0}^{\infty} P(i+1,1,1,j,O) + \lambda P(i-1,1,1,0,I) \tag{3.2e}$$

(f) For $i = 0, j = 0, e_1 = 1, e_2 = 0$.

$$(\lambda + \mu_1 + mu_2)P(0,1,1,0,I) = \mu_2 \sum_{j=0}^{\infty} P(1,1,1,j,O) + \lambda P(0,1,0,0,I) \tag{3.2f}$$

(g) For $i = 0, j = 0, e_1 = 1, e_2 = 0$.

$$(\lambda + \mu_1)P(0,1,0,0,I) = \mu_2 \sum_{j=0}^{\infty} P(0,1,1,j,O) + \lambda P(0,0,0.0) \tag{3.2g}$$

2. The equilibrium equations for the states for which $Z = O$

(a) For $0 < i < B, j > 0, e_1 = 1, e_2 = 1$.

$$(\lambda + \mu_1 + \mu_2)P(i,1,1,j,O) = \mu_1 P(i+1,1,1,j-1,O) + \lambda P(i-1,1,1,j,O) \tag{3.3a}$$

(b) For $i = B, j \geq 0, e_1 = 1, e_2 = 1$.

$$(\mu_1 + \mu_2)P(B,1,1,j,O) = \lambda P(B-1,1,1,j,O) \tag{3.3b}$$

(c) For $i = 0, j > 0, e_1 = 1, e_2 = 1$.

$$(\lambda + \mu_1 + \mu_2)P(0,1,1,j,O) = \mu_1 P(1,1,1,j-1,O) + \lambda P(0,0,1,j,O) \tag{3.3c}$$

(d) For $i = 0, j > 0, e_1 = 0, e_2 = 1$.

$$(\lambda + \mu_2)P(0,0,1,j,O) = \mu_1 P(0,1,1,j-1,O) \tag{3.3d}$$

(e) For $0 < i < B, j = 0, e_1 = 1, e_2 = 1$.

$$(\lambda + \mu_1 + \mu_2)P(i,1,1,0,O) = \mu_1 \sum_{j=0}^{\infty} P(i+1,1,1,j,O) + \lambda P(i-1,1,1,0,O) \tag{3.3e}$$

(f) For $i = 0, j = 0, e_1 = 1, e_2 = 1$.

$$(\lambda + \mu_1 + \mu_2)P(0,1,1,0,O) = \mu_1 \sum_{j=0}^{\infty} P(1,1,1,j,I) + \lambda P(0,0,1,0,O) \tag{3.3f}$$

(g) For $i = 0, j = 0, e_1 = 0, e_2 = 1$.

$$(\lambda + \mu_2)P(0,0,1,0,O) = \mu_1 \sum_{j=0}^{\infty} P(0,1,1,j,I) \tag{3.3g}$$

4. The Case $B = 0$

Explicit closed form expressions can be obtained for the buffer occupation probabilities for the special case when $B = 0$.

A customer who arrives when both the servers are busy is discarded. Because of the resequencing constraint, customers leave the system in the same order in which they started service. We assume that the resequencing box has unlimited buffer space.

Note that all the results given below can be recovered from the more general discussion of Section 5. We nevertheless go through the calculations because the case $B = 0$ is of interest in its own right and the equations being much simpler than for the general case, it serves an illustrative purpose.

The equations to be solved are now stated below. Since $n = 0$ everywhere, it is omitted from the notation. The equations (4.1)-(4.3) now become,

$$\lambda P(0) = \mu_1 \sum_{j=0}^{\infty} P(1,0,j,I) + \mu_2 \sum_{j=0}^{\infty} P(0,1,j,O) \quad j = 0,1\ldots \tag{4.4}$$

$$(\lambda + \mu_1)P(1,0,j,I) = \mu_2 P(1,1,j-1,I) \qquad j = 1,2\ldots \tag{4.5a}$$

$$(\lambda + \mu_2)P(0,1,j,O) = \mu_1 P(1,1,j-1,O) \qquad j = 1,2\ldots \tag{4.5b}$$

$$(\mu_1 + \mu_2)P(1,1,j,I) = \lambda P(1,0,j,I) \qquad j = 0,1\ldots \tag{4.6a}$$

$$(\mu_1 + \mu_2)P(1,1,j,O) = \lambda P(0,1,j,O) \qquad j = 0,1\ldots \tag{4.6b}$$

$$(\lambda + \mu_1)P(1,0,0,I) = \lambda P(0) + \mu_2 \sum_{j=0}^{\infty} P(1,1,j,O) \tag{4.7a}$$

$$(\lambda + \mu_2)P(0,1,0,O) = \mu_1 \sum_{j=0}^{\infty} P(1,1,j,I) \tag{4.7b}$$

We now proceed to solve these equations. From (4.5a-b) and (4.6a-b) it is easy to see that the relations

$$P(1,0,j,I) = (\frac{\mu_2}{\lambda + \mu_1})^j (\frac{\lambda}{\mu_1 + \mu_2})^j P(1,0,0,I) \qquad j = 0,1\ldots \tag{4.8}$$

$$P(1,1,j,I) = (\frac{\mu_2}{\lambda + \mu_1})^j (\frac{\lambda}{\mu_1 + \mu_2})^{j+1} P(1,0,0,I) \qquad j = 0,1\ldots \tag{4.9}$$

$$P(0,1,j,O) = (\frac{\mu_1}{\lambda + \mu_2})^j (\frac{\lambda}{\mu_1 + \mu_2})^j P(0,1,0,O) \qquad j = 0,1\ldots \tag{4.10}$$

$$P(1,1,j,O) = (\frac{\mu_1}{\lambda + \mu_2})^j (\frac{\lambda}{\mu_1 + \mu_2})^{j+1} P(0,1,0,O) \qquad j = 0,1\ldots \tag{4.11}$$

are satisfied.

Substituting (4.11) into (4.7a), we obtain

$$(\lambda+\mu_1)P(1,0,0,I) = \lambda P(0) + \mu_2 \sum_{j=0}^{\infty} (\frac{\mu_1}{\lambda+\mu_2})^j (\frac{\lambda}{\mu_1+\mu_2})^{j+1} P(0,1,0,O) \qquad (4.12)$$

Pose

$$\sigma_1 = \sum_{j=0}^{\infty} (\frac{\mu_1}{\lambda+\mu_2})^j (\frac{\lambda}{\mu_1+\mu_2})^{j+1}$$

$$= \frac{\lambda(\lambda+\mu_2)}{\mu_2(\lambda+\mu_1+\mu_2)}$$

with σ_1 always finite since

$$\frac{\mu_1}{\lambda+\mu_2} \frac{\lambda}{\mu_1+\mu_2} < 1$$

Hence (4.12) can be rewritten as

$$(\lambda+\mu_1)P(1,0,0,I) = \lambda P(0) + \sigma_1 \mu_2 P(0,1,0,O) \qquad (4.13)$$

Substituting (4.9) into (4.7b), we also obtain

$$(\lambda+\mu_2)P(0,1,0,O) = \mu_1 \sum_{j=0}^{\infty} (\frac{\mu_2}{\lambda+\mu_1})^j (\frac{\lambda}{\mu_1+\mu_2})^{j+1} P(1,0,0,I) \qquad (4.14)$$

Pose

$$\sigma_2 = \sum_{j=0}^{\infty} (\frac{\mu_2}{\lambda+\mu_1})^j (\frac{\lambda}{\mu_1+\mu_2})^{j+1}$$

$$= \frac{\lambda(\lambda+\mu_1)}{\mu_1(\lambda+\mu_1+\mu_2)}$$

with σ_2 obviously finite since

$$\frac{\mu_2}{\lambda+\mu_1}\frac{\lambda}{\mu_1+\mu_2} < 1.$$

Hence (4.14) can be rewritten as

$$(\lambda+\mu_2)P(0,1,0,O) = \sigma_2\mu_1 P(1,0,0,I) \tag{4.15}$$

The relations (4.13) and (4.15) provide us with two equations for the unknown values of $P(0), P(1,0,0,I)$ and $P(0,1,0,O)$. In order to get a third equation, we use the fact that the sum of all the probabilities should be one, i.e.,

$$P(0) + \sum_{j=0}^{\infty}(P(1,0,j,I) + P(1,1,j,I) + P(0,1,j,O) + P(1,1,j,O)) = 1 \tag{4.16}$$

Substituting from (4.8)-(4.11) into (4.16), we obtain

$$P(0) + (\sigma_2+\sigma_3)P(1,0,0,I) + (\sigma_1+\sigma_4)P(0,1,0,O) = 1 \tag{4.17}$$

where σ_1 and σ_2 are as defined earlier and σ_3 and σ_4 are given by

$$\sigma_3 = \sum_{j=0}^{\infty}(\frac{\mu_2}{\lambda+\mu_1})^j(\frac{\lambda}{\mu_1+\mu_2})^j$$

$$= \frac{(\lambda+\mu_1)(\mu_1+\mu_2)}{\mu_1(\lambda+\mu_1+\mu_2)}$$

and

$$\sigma_4 = \sum_{j=0}^{\infty}(\frac{\mu_1}{\lambda+\mu_2})^j(\frac{\lambda}{\mu_1+\mu_2})^j$$

$$= \frac{(\lambda+\mu_2)(\mu_1+\mu_2)}{\mu_2(\lambda+\mu_1+\mu_2)}.$$

The values of $P(0), P(1,0,0,I)$ and $P(0,1,0,O)$ can now be obtained very easily by solving the system of linear equations

$$\begin{pmatrix} \lambda & -(\lambda+\mu_1) & \sigma_1\mu_2 \\ 0 & \sigma_2\mu_1 & -(\lambda+\mu_2) \\ 1 & (\sigma_2+\sigma_3) & (\sigma_1+\sigma_4) \end{pmatrix} \begin{pmatrix} P(0) \\ P(1,0,0,I) \\ P(0,1,0,O) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.18}$$

With

$$\psi := \frac{(\lambda+\mu_1)(\lambda+\mu_2)}{\sigma_2\mu_1\lambda} - \frac{\sigma_1\mu_2}{\lambda} + \frac{(\sigma_2+\sigma_3)(\lambda+\mu_2)}{\sigma_2\mu_1} + \sigma_1 + \sigma_4 \tag{4.19}$$

routine yet tedious calculations show that

$$P(0) = \frac{(\lambda+\mu_1)(\lambda+\mu_2)}{\sigma_2\mu_1\lambda\psi} - \frac{\sigma_1\mu_2}{\lambda\psi} \tag{4.20}$$

$$P(1,0,0,I) = \frac{(\lambda+\mu_2)}{\sigma_2\mu_1\psi} \tag{4.21}$$

$$P(0,1,0,O) = \frac{1}{\psi} \tag{4.22}$$

We can use (4.8)-(4.11) to obtain the values the other steady state probabilities.

1) The probability that there are j customers in the resequencing buffer and the customer who arrived earlier is being served by the fast server.

$$P(j,I) = (\frac{\lambda+\mu_1}{\mu_1+\mu_2})\frac{\lambda+\mu_2}{\sigma_2\mu_1\psi}(\frac{\mu_2}{\lambda+\mu_1})^j(\frac{\lambda}{\mu_1+\mu_2})^j \qquad j = 0, 1 \ldots (4.23)$$

2) The probability that there are j customers in the resequencing buffer and the customer who has arrived earlier is being served by the slow server.

$$P(j,O) = \frac{\lambda+\mu_1+\mu_2}{\mu_1+\mu_2}\frac{1}{\psi}(\frac{\mu_1}{\lambda+\mu_2})^j(\frac{\lambda}{\mu_1+\mu_2})^j \qquad j = 0, 1 \ldots (4.24)$$

3) The probability q_j that there are j customers in the resequencing buffer.

$$q_j = \begin{cases} P(0) + \frac{\lambda+\mu_1+\mu_2}{\mu_1+\mu_2}\left[\frac{\lambda+\mu_2}{\sigma_2\mu_1}+1\right]\frac{1}{\psi} & \text{if } j=0 \\ (\frac{\lambda}{\mu_1+\mu_2})^j \frac{\lambda+\mu_1+\mu_2}{\mu_1+\mu_2}\left[(\frac{\mu_2}{\lambda+\mu_1})^j \frac{\lambda+\mu_2}{\sigma_2\mu_1} + (\frac{\mu_1}{\lambda+\mu_2})^j\right]\frac{1}{\psi} & \text{for } j=1,2\ldots \end{cases} \tag{4.25}$$

5. The General Case

In the present section we present a technique for calculating the exact values of the buffer occupation probabilities in the $M/M/2/B$ queue with resequencing. Note that Eqns. (4.8)-(4.11) in the last section indicate a geometric structure for the buffer occupation probabilities when B=0. We carry that insight to its logical conclusion by showing that in the general case, the buffer occupation probabilities have a *matrix-geometric* structure.

We proceed as follows. The states in the Markov chain are numbered appropriately so that the corresponding infinitesmal generator matrix Q is seen to have matrix-geometric structure. In fact the structure coincides with the modified matrix associated with complex boundary behavior identified by Neuts in [8, p.24]. Once this is done, the probability vector can be written down using standard techniques.

As the first step we partition the state probability vector into the vectors $(P(0), \pi_0, \pi_1, \ldots)$, where $P(0)$ is the probability of the zero state as before and

$$\pi_j = (P(0,0,1,j,O), P(0,1,1,j,O), \ldots, P(B,1,1,j,O), P(B,1,1,j,I), \ldots$$

$$\ldots, P(0,1,1,j,I), P(0,1,0,j,I)) \qquad j = 0,1\ldots \tag{4.26}$$

Hence π_j is a $(1 \times 2(B+2))$ row vector which contains the probabilities of all states that have j customers in the resequencing buffer. Using this partition of the state probability vector, we can write the infinitesmal generator matrix Q in the block partition form

$$Q = \begin{pmatrix} D0 & C0 & 0 & 0 & 0 & 0 & \dots \\ D1 & C1 & A0 & 0 & 0 & 0 & \dots \\ D1 & C2 & A1 & A0 & 0 & 0 & \dots \\ D1 & C2 & 0 & A1 & A0 & 0 & \dots \\ D1 & C2 & 0 & 0 & A1 & A0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{4.27}$$

In (4.27), $D0 = -\lambda$ and the other matrices are defined below, with the convention $\gamma = (\lambda + \mu_1 + \mu_2)$, by

$$C0 = (0, 0, \dots, 0, \lambda)^{1\times 2(B+2)}$$

$$D1^T = (\mu_2, 0, \dots, 0, \mu_1)^{1\times 2(B+2)}$$

$$A0 = \begin{pmatrix} A0_{11} & 0 \\ 0 & A0_{22} \end{pmatrix}$$

where

$$A0_{11} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \mu_1 & 0 \end{pmatrix}^{(B+2)\times(B+2)}$$

and

$$A0_{22} = \begin{pmatrix} 0 & \mu_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}^{(B+2)\times(B+2)},$$

$$A1 = \begin{pmatrix} A1_{11} & 0 \\ 0 & A1_{22} \end{pmatrix}$$

where

$$A1_{11} = \begin{pmatrix} -(\lambda+\mu_2) & \lambda & 0 & \dots & 0 & 0 & 0 \\ 0 & -\gamma & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -\gamma & \lambda \\ 0 & 0 & 0 & \dots & 0 & 0 & -(\mu_1+\mu_2) \end{pmatrix}^{(B+2)\times(B+2)}$$

and

$$A1_{22} = \begin{pmatrix} -(\mu_1+\mu_2) & 0 & 0 & \dots & 0 & 0 & 0 \\ \lambda & -\gamma & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -\gamma & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda & -(\lambda+\mu_1) \end{pmatrix}^{(B+2)\times(B+2)} ,$$

$$C2 = \begin{pmatrix} 0 & C2_{12} \\ C2_{21} & 0 \end{pmatrix}$$

where

$$C2_{12} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & \dots & 0 & \mu_2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \mu_2 & \dots & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}^{(B+2)\times(B+2)}$$

and

$$C2_{21} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \dots & \mu_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \mu_1 & 0 & \dots & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}^{(B+2)\times(B+2)} ,$$

and

$$C1 = \begin{pmatrix} C1_{11} & C1_{12} \\ C1_{21} & C1_{22} \end{pmatrix}$$

where

$$C1_{11} = A1_{11}, \quad C1_{12} = C2_{12}, \quad C1_{21} = C2_{21} \text{ and } C1_{22} = A1_{22}$$

Let e be a $2(B+2) \times 1$ column vector with all its components equal to one. Since Q is an infinitesmal generator matrix, its rows should sum upto zero,i.e.,

$$\begin{aligned} D0 + C0e &= 0 \\ D1 + C1e + A0e &= 0 \\ D1 + C2e + A1e + A0e &= 0 \end{aligned} \tag{4.28}$$

We now proceed with the task of solving the equations

$$\pi Q = 0, \qquad \pi e = 1 \tag{4.29}$$

which can be rewritten as

$$P(0)D0 + D1 \sum_{i=0}^{\infty} \pi_i = 0 \tag{4.30}$$

$$P(0)C0 + \pi_0 C1 + C2 \sum_{i=1}^{\infty} \pi_i = 0 \tag{4.31}$$

$$\pi_i A0 + \pi_{i+1} A1 = 0 \qquad i \geq 0 \tag{4.32}$$

$$P(0) + \sum_{i=0}^{\infty} \pi_i = 1 \tag{4.33}$$

Before we can solve (4.30)-(4.33) we need the following

Lemma 1. *The following statements hold true, namely*

(1) The matrix $A1$ *is nonsingular.*

(2) If

$$R = -A0(A1^{-1}) \tag{4.34}$$

then the eigenvalue $\lambda(R)$ *of* R *with largest modulus satisfies the condition,*

$$\lambda(R) < 1. \tag{4.35}$$

(3) The matrix $B(R)$ *defined by*

$$B(R) = \begin{pmatrix} D0 & C0 \\ (I-R)^{-1}D1 & C1 + R(I-R)^{-1}C2 \end{pmatrix} \tag{4.36}$$

is an infinitesmal generator matrix.

Proof. (1) The nonsingularity of $A1$ can be proved very easily as follows. If the row vector $u = (u_1, u_2)$ is in the (left) null space of $A1$, then

$$uA1 = 0 \tag{4.37}$$

and this implies that

$$(\lambda + \mu_2)u_1 = 0 \qquad \text{and} \quad (\lambda + \mu_1)u_2 = 0 \tag{4.38}$$

whence $u_1 = u_2 = 0$, i.e., $A1$ is nonsingular.

(2) We will prove (4.35) by using Theorem A from the appendix. Note that R can written as

$$R = \begin{pmatrix} -A0_{11}A1_{11}^{-1} & 0 \\ 0 & -A0_{22}A1_{22}^{-1} \end{pmatrix}$$

Denoting $R_{11} = -A0_{11}A1_{11}^{-1}$ and $R_{22} = -A0_{22}A1_{22}^{-1}$, it is sufficient to show that $\lambda(R_{11}) < 1$ and $\lambda(R_{22}) < 1$. We will show that $\lambda(R_{11}) < 1$ and leave the proof of the other claim to the interested reader. We now write down the matrix R_{11} explicitly by substituting for $A0_{11}$ and $A1_{11}$. We let $\delta = \mu_1 + \mu_2$ and $\eta = \lambda + \mu_2$ in what follows, so that

$$R_{11} = \frac{1}{\delta\eta\gamma^B}\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mu_1\gamma^B\delta & \mu_1\lambda\gamma^{B-1}\delta & \mu_1\lambda^2\gamma^{B-2}\delta & \dots & \mu_1\lambda^B\delta & \mu_1\lambda^{B+1} \\ 0 & \mu_1\gamma^{B-1}\delta\eta & \mu_1\lambda\gamma^{B-2}\delta\eta & \dots & \mu_1\lambda^{B-1}\delta\eta & \mu_1\lambda^B\eta \\ 0 & 0 & \mu_1\gamma^{B-1}\delta\eta & \dots & \mu_1\lambda^{B-2}\gamma\delta\eta & \mu_1\lambda^{B-1}\eta \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu_1\gamma^{B-1}\delta\eta & \mu_1\lambda\gamma^{B-1}\eta \end{pmatrix}.$$

Since the top row of R_{11} consists only of zeroes, it follows that 0 is an eigenvalue of R_{11}. The remaining $(B+1)$ eigenvalues come from the $(B+1)\times(B+1)$ matrix obtained by omitting the first row and the first column. It can be shown that the elements of the first row of this matrix sum up to $\frac{\lambda\mu_1}{(\mu_1+\mu_2)(\lambda+\mu_2)}$, while the remaining rows sum up to $\frac{\mu_1}{(\mu_1+\mu_2)}$. Hence the criteria of Theorem A from the appendix is satisfied, and it follows that $\lambda(R_{11}) < 1$.

(3) Since $\lambda(R) < 1$, it follows that $\sum_{i=0}^{\infty} R^i = (I-R)^{-1}$ is well defined. To prove that $B(R)$ is an infinitesmal generator, first note that $D0 + C0e = 0$ by (4.28). Hence it suffices to verify that

$$\sum_{i=1}^{\infty} R^i(D1 + C2e) + D1 + C1e = 0. \tag{4.39}$$

Substituting from (4.28) for $D1, C1$ and $C2$, we see that (4.39) is equivalent to

$$R(I-R)^{-1}(A1 + A0)e + A0e = 0$$

i.e.,

$$A0(A1)^{-1}(I + A0(A1)^{-1})(A1 + A0)e = A0e$$

upon using (4.34). We finally obtain

$$A0(A1)^{-1}(I + A0(A1)^{-1})(I + A0(A1)^{-1})A1e = A0e$$

This verifies (4.39).

∎

We can now state the main result in this section.

Theorem 1. *The solution to (4.29) is given by the vector* $\pi = (P(0), \pi_0, \pi_1, \ldots)$ *where*

$$\pi_i = \pi_0 R^i, \qquad i \geq 0 \tag{4.40}$$

with R *defined by (4.34) and* $(P(0), \pi_0)$ *solves the equation*

$$(\, P(0) \quad \pi_0 \,)\, B(R) = 0 \tag{4.41}$$

subject to the normalization condition

$$P(0) + \pi_0 (I - R)^{-1} e = 1 \tag{4.42}$$

Proof. By Lemma 1, A1 is nonsingular, so that (4.40) follows directly from (4.32). Also (4.41) follows from (4.30)-(4.31) after substituting for $\{\pi_i, i \geq 1\}$ in terms of π_0 via (4.40).

∎

The distribution of the number of customers in the resequencing buffer can be recovered from Theorem 1. From the definition of π_i given in (4.26), the probability q_j of finding j customers in the resequencing buffer is simply the sum of the probabilities in π_j, therefore,

$$q_j = \begin{cases} P(0) + \pi_0 e & \text{if } j = 0 \\ \pi_0 R^j e & \text{for } j = 1, 2 \ldots \end{cases} \tag{4.43}$$

The average number of customers in the resequencing buffer is then given by

$$\begin{aligned} \overline{N} &= \sum_{j=1}^{\infty} j\, q_j \\ &= \pi_0 \sum_{j=1}^{\infty} j\, R^j e \\ &= \pi_0 R(I - R)^{-2} e. \end{aligned} \tag{4.44}$$

6. Numerical Results

In this section we give an application of the formulae that were derived in the last section. Specifically, we are interested in obtaining values for the resequencing buffer size $N_{0.05}$ subject to the constraint that probability that it overflows is equal to or less than 0.05. We shall consider the special cases $B = 0$ and $B = 1$.

The Case B=0

We carried out the calculations for the following values of the arrival and service rates $\lambda = 1, \mu_1 = 1$ and $\mu_2 = 0.1$. Substituting these values of λ, μ_1 and μ_2 in (4.20) and (4.25), we obtain

$$q_0 = 0.3994$$

and

$$q_j = 0.126\left[1.16(0.045)^j + (0.83)^j\right]. \qquad j = 1, 2 \ldots$$

Using this formula, it can be shown that

$$N_{0.05} = 10 \quad \text{and} \quad \overline{N} = 3.62$$

The Case B=1

Once again, we carried out the calculations for the following valued of the arrival and service rates $\lambda = 1, \mu_1 = 1$ and $\mu_2 = 0.1$. For this case we use (4.13) to obtain the values of $q_j, j = 0, 1 \ldots$. The required matrix manipulations were carried out with the help of the PROMATLAB mathematical software package. We found out that

$$\overline{N} = 5.03$$

while the values of q_j are given in the following table.

j	q_j	j	q_j
0	0.3187	10	0.0214
1	0.0788	11	0.0185
2	0.0678	12	0.0161
3	0.0587	13	0.0139
4	0.0508	14	0.0120
5	0.0440	15	0.0104
6	0.0381	16	0.0090
7	0.0330	17	0.0078
8	0.0286	18	0.0068
9	0.0247	19	0.0056

From this table, it is easy to see that $N_{0.05} = 18$.

We notice that $N_{0.05}$ increases from 10 to 18 as B is increased from 0 to 1. This can be understood due to the fact that the system is in heavy traffic (since $\lambda = 0.1$ and $\mu_1 + \mu_2 = 1.1$), so that the extra buffer for the case $B = 1$ remains full most of the time. This leads to a larger supply of customers to the two servers, leading to more customers going out of sequence.

Acknowledgements: This work was done as part of the authors MS thesis. The author would like to acknowledge the advice and help he got from his advisor Dr. Armand Makowski. The author would also like to thank Dr. Levent Gun for suggesting the use of Theorem A in order to prove Lemma 1.

APPENDIX

The following theorem is stated without proof, the interested reader may consult [10, Cor. 6.6, p.227].

Theorem A. *Consider a square matrix A with components* $a_{ij}, i, j = 1 \ldots n$. *If the sums*

$$\sum_{j=1}^{n} | a_{ij} |, \qquad 1 \leq i \leq n \quad (A1)$$

are all less than 1, *or if*

$$\sum_{i=1}^{n} | a_{ij} |, \qquad 1 \leq i \leq n \quad (A2)$$

are all less than 1, *then all the eigenvalues of* A *are inside the unit circle.*

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