COMPARISON OF SCHEDULING STRATEGIES

IN MULTI–PROCESSOR SYSTEMS

by

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ABSTRACT

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I. INTRODUCTION

In this paper we compare the performance of two multiple processor queueing structures, namely the Fork Join queue and a system of parallel queues with Bernoulli routing. Both systems are assumed to have K (≥ 2) identical servers operating in *parallel* with infinite waiting rooms. Jobs that arrive to these systems are assumed to consist of exactly K tasks, the service requirements of the tasks being independent and identically distributed (i.i.d.).

Upon arrival into the Fork–Join system, a job is instantaneously decomposed into its K constituent tasks and the k^{th} task is routed to the k^{th} queue where it is served in FCFS order. As soon as a task completes service, it is put into a synchronization buffer, and a job leaves the system when all of its constituent tasks have completed service. In the system of parallel queues with Bernoulli routing, an arriving job is routed to the k^{th} queue with probability $\frac{1}{K}$, $1 \leq k \leq K$, with the routing decision being independent of any other event, past, present or future. In each queue, jobs are processed in a FCFS manner.

The Fork–Join queue has been proposed as a queueing model for parallel processing: an approximate analysis of the job response time was given by Nelson and Tantawi [NT] when the arrival process is Poisson, task service times are exponentially distributed, and all the servers are identical. A more complete analysis that does not restrict all processors to have the same speed can be found in Kim and Agarwala [KA]. In the special case when $K = 2$, and the interarrival and service times are exponentially distributed, Flatto and Hahn [FH] have determined the stationary joint distribution of the number of customers in each of the queues. In Nelson, Tantawi and Towsley [NTT], a comparison of various queueing models of parallel processing is carried out, and it is shown that with Poisson arrivals and exponentially distributed task service times, the mean response time of a job in the Fork–Join queue is less than the mean job response time in three closely related queueing systems. In particular, the Fork–Join queue was shown have a lower mean job response time than the system of parallel queues with Bernoulli routing. The interested reader is referred to [NTT] for further details. Duda and Czachorski [DC] use the Fork–Join queue to model the ParBegin and ParEnd constructs of parallel programming languages, and analyze the performance of parallel programs that use these constructs using a closed queueing model. In related work, Kanakia and Tobagi [KT] examine parallel programs that are constructed from two kinds of fork and join structures (If–Then–Else and Fork–Join), and analyze their running time on uniproccessors and distributed systems under various processor scheduling policies. Baccelli, Makowski and Shwartz [BMSa] analyze the Fork–Join queue when the interarrival and service times are arbitrarily distributed, and in contrast to the previous references, they explore the *structural* properties of the Fork–Join queue. In particular, they develop a lower bound for the job response time that holds in the increasing convex ordering, and an upper bound that holds in the strong stochastic ordering. In [BMSb] they also present approximations for the job response times, and compare the approximation to simulations.

This paper concerns itself with the comparison of these two queueing systems and is organized as follows. In Section II we formalize the models of the Fork–Join queue and the system of parallel queues with Bernoulli routing and state all the assumptions we make. In Section III we show that the task response times in the Fork–Join queue are smaller, in the convex increasing ordering, than the task response times in the system of parallel queues with Bernoulli routing. In Section IV, we show that the job response time of the n^{th} customer in the Fork–Join queue is increasing and concave in K. In section V we examine the system of parallel queues with Bernoulli routing and show that the response time of the n^{th} customer is bounded below by a function that is convex in K, so that for sufficiently large K the job response time of the n^{th} customer in the Fork–Join queue will be smaller, in the convex increasing ordering, than the job response time in the system of queues with Bernoulli routing. The value of K at which this crossover occurs depends on the utilization of the system. In Section VI, by examining a heavy traffic diffusion limit we show that this is essentially the best possible result, i.e. if the arrival process is assumed to be Poisson, for any fixed value of K the ratio of the steady state response time in the two systems can be greater or less than 1, depending on the coefficient of variation of the task service time. If the coefficient of variation is sufficiently small, the mean job response time in the Fork Join system is smaller than that in the system with Bernoulli routing. When the coefficient of variation is large, the system with Bernoulli routing has a lower job response time. In Section VII, we examine the two systems in light traffic and show that there is some $\rho_0(K) > 0$ such that for all utilizations strictly less than $\rho_0(K)$ and for all $K \geq 2$ the response time of the n^{th} customer in the Fork Join system is less than that of the n^{th} customer in the system of queues with Bernoulli routing. Finally, in Section VIII, we summarize our findings and discuss some open problems.

II. PRELIMINARY NOTATION

We follow the notation used in (Baccelli and Makowski [1]) and (Shaked and Shantikumar [11,12]). The set of real (resp. non–negative real) numbers is denoted by $I\!R$ (resp. $I\!R_+$). The k^{th} component of any element x in $I\!R^K$ is denoted by x^k, $1 \leq k \leq K$. For any two vectors x and y in $I\!R^K$, the ordering $x \leq y$ is interpreted componentwise to read $x^k \leq y^k$, $1 \leq k \leq K$. A mapping $f : I\!R^K \rightarrow I\!R$ is then said to be increasing (resp. decreasing) if $x \leq y$ in $I\!R^K$ implies $f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$).

We find it convenient to define all the random variables (rvs) of interest on some common probability triple $(\Omega, \mathcal{F}, P)$. A probability distribution function F on $I\!R^K$ is routinely identified with an $I\!R^K$–valued rv $X = (X^1, \ldots, X^K)$ which has distribution F, in which case

$$F(x) = P[X^1 \leq x^1, \ldots, X^K \leq x^K], \quad x \in I\!R^K. \tag{2.1}$$

Two $I\!R^K$–valued rvs X and Y are then said to be *equal in law* if they have the same distribution, a fact we denote by $X =_{st} Y$. Moreover, for any $I\!R$–valued rv X with distribution function F, we denote its mean and variance by $m(X)$ and $\mathrm{var}(X)$, respectively, whenever these quantities are well defined; the notation $m(F)$ and $\mathrm{var}(F)$ is used interchangeably for $m(X)$ and $\mathrm{var}(X)$.

For any sequence of $I\!R^K$–valued rvs $\{X_n,\ n = 0, 1, \ldots\}$, we denote its weak limit (as n goes to infinity) by X_∞ whenever it exists, i.e., X_∞ is any $I\!R^K$–valued rv with the property that

$$P[X_\infty \leq x] = \lim_n P[X_n \leq x] \tag{2.2}$$

for all x in $I\!R^K$ which are points of continuity for the distribution of X_∞. We call X_∞ the stationary (or steady–state) version of the sequence $\{X_n,\ n = 0, 1, \ldots\}$.

The reader is referred to the monographs by Ross [10] and Stoyan [14] for additional information and properties of the orderings $\leq_{st}$ and $\leq_{icx}$ on the collection $\mathcal{D}(\mathbb{R}^K)$ of probability distribution functions on $\mathbb{R}^K$.

III. MODELS: NOTATION AND ASSUMPTIONS

In order to facilitate the comparison between the two queueing systems, we construct them from the same set of basic rvs. With this in mind, we start with the integrable $\mathbb{R}_+$-valued rvs $\{\tau_{n+1},\ n = 0, 1, \ldots\}$, the integrable $\mathbb{R}_+^K$-valued rvs $\{\sigma_n,\ n = 0, 1, \ldots\}$, and the $\{1, \ldots, K\}$-valued rvs $\{\nu_n,\ n = 0, 1, \ldots\}$. With this last sequence we associate a new sequence of $\{0, 1\}^K$-valued rvs $\{u_n,\ n = 0, 1, \ldots\}$ by setting

$$u_n^k := \delta(\nu_n, k), \quad 1 \leq k \leq K \qquad n = 0, 1 \ldots (2.1)$$

with $\delta(i, j) = 1$ if $i = j$ and $\delta(i, j) = 0$ if $i \neq j$. We also need to introduce the $\mathbb{R}_+$-valued rvs $\{\bar{\sigma}_n,\ n = 0, 1, \ldots\}$ which are defined by

$$\bar{\sigma}_n := \sum_{k=1}^{K} \sigma_n^k. \qquad n = 0, 1 \ldots (2.2)$$

These quantities are interpreted as follows: The interarrival time between the n^{th} and the $(n+1)^{rst}$ jobs is given by τ_{n+1} with the convention that the 0^{th} job arrives at time $t = 0$. The n^{th} job consists of K independent tasks; the execution of the k^{th} task from the n^{th} job requires σ_n^k units of time, $1 \leq k \leq K$, so that the total execution time of the n^{th}job is exactly $\bar{\sigma}_n$. In the system of parallel queues with Bernoulli routing, $\nu_n = k$ (or equivalently $u_n^k = 1$) indicates that the n^{th} job joins the k^{th} queue.

In each case we define the performance measures of interest under the simplifying assumption that the system is empty at time $t = 0$:

For the Fork–Join queue, the $\mathbb{R}_+^K$-valued rvs $\{W_n,\ n = 0, 1, \ldots\}$ are generated componentwise by the recursion

$$W_{n+1}^k = \left[W_n^k + \sigma_n^k - \tau_{n+1} \right]^+, \quad 1 \leq k \leq K \qquad n = 0, 1, \ldots (2.3)$$

$$W_0^k = 0,$$

where W_n^k represents the waiting time of the k^{th} task from the n^{th} job. The corresponding response time R_n^k (through the k^{th} channel) is thus

$$R_n^k := W_n^k + \sigma_n^k, \quad 1 \le k \le K. \qquad n = 0, 1 \ldots (2.4)$$

The system response time T_n of the n^{th} job is then given by

$$T_n = \max_{1 \le k \le K} R_n^k. \qquad n = 0, 1 \ldots (2.5)$$

For the system of parallel queues with Bernoulli routing, we generate the $I\!R_+^K$–valued rvs $\{V_n,\ n = 0, 1, \ldots\}$ componentwise by the recursion

$$\begin{aligned} V_{n+1}^k &= \left[V_n^k + u_n^k \bar{\sigma}_n - \tau_{n+1} \right]^+, \quad 1 \le k \le K \\ V_0^k &= 0. \end{aligned} \qquad n = 0, 1, \ldots (2.6)$$

We take the view that each job brings work to *every* queue but that only the work executed by the processor of the queue which the job joins has (possibly) non–zero service duration. Clearly, V_n^k represents the work (expressed in remaining processing time) present in the k^{th} queue as the n^{th} job enters the system. In other words, V_n^k is the amount of time it would have to wait in the queue before receiving service if it were assigned to the k^{th} processor, i.e., if $u_n^k = 1$. The waiting time U_n and the system response time S_n of the n^{th} job are thus given by

$$U_n = \sum_{k=1}^{K} u_n^k V_n^k \qquad n = 0, 1 \ldots (2.7)$$

and

$$S_n = \sum_{k=1}^{K} u_n^k \cdot \left(V_n^k + \bar{\sigma}_n \right), \qquad n = 0, 1 \ldots (2.8)$$

respectively.

We are interested in comparing the system response times T_n and S_n, either in transient or in statistical equilibrium. Throughout this discussion, we assume conditions **(A1)–(A4)** to hold, namely

(A1): The sequences $\{\tau_{n+1},\ n=0,1,\ldots\}$, $\{\sigma_n,\ n=0,1,\ldots\}$ and $\{\nu_n,\ n=0,1,\ldots\}$ are *mutually independent*;

(A2): The $\mathbb{R}_+$–valued rvs $\{\tau_{n+1},\ n=0,1,\ldots\}$ form an *i.i.d.* sequence with common distribution A;

(A3): The $\mathbb{R}_+$–valued rvs $\{\sigma_n^k,\ 1\le k\le K,\ n=0,1,\ldots\}$ form an *i.i.d.* sequence with common distribution B; and

(A4): The $\{1,\ldots,K\}$–valued rvs $\{\nu_n,\ n=0,1,\ldots\}$ form an *i.i.d.* sequence with common distribution

$$P[\nu_n=k]=\frac{1}{K},\quad 1\le k\le K. \qquad n=0,1\ldots(2.9)$$

For future reference, we point out that under the assumptions **(A1)**–**(A4)**, we have

$$V_n^1=_{st}\ldots=_{st}V_n^k=_{st}U_n \qquad n=0,1\ldots(2.10)$$

as the system is symmetric, so that

$$S_n=_{st}U_n+\bar{\sigma}_n. \qquad n=0,1\ldots(2.11)$$

It is well known that the stability condition of both systems is given by

$$E(B)<E(A). \tag{2.12}$$

Wer ecall that under (2.12), in each of the queueing systems under consideration, the sequences of waitng times and response times have stationary versions. We also note that this stability condition (2.12) does not depend on K, the number of processors, and is equivalent to

$$\rho:=\frac{E[B]}{E[A]}<1. \tag{2.13}$$

Since ρ is the utilization of a single processor in either queueing system, we see that the comparison attempted here is indeed a fair one.

Throughout this discussion, we shall use the superscript (K) in the notation to indicate that the quantities of interest are defined for systems with K parallel servers. It is convenient to view the service times as a two–dimensional array of i.i.d. $\mathbb{R}_+$–valued rvs $\{\sigma_n^k,\ k = 1, 2, \ldots, K; n = 0, 1, \ldots\}$ with common distribution B.

III. A BASIC COMPARISON RESULT

Our first comparison result is contained in

Theorem 3.1. *Under the assumptions* **(A1)–(A4)**, *we have the stochastic ordering relations*

$$W_n^k \leq_{icx} V_n^k, \quad 1 \leq k \leq K \qquad n = 0, 1 \ldots (3.1)$$

and

$$R_n^k \leq_{icx} U_n + \sum_{i=1}^{k} \sigma_n^i, \quad 1 \leq k \leq K. \qquad n = 0, 1 \ldots (3.2)$$

Proof. Invoking Proposition C.1 from Appendix C, we note that

$$\sigma_n^k \leq_{icx} u_n^k \cdot \bar{\sigma}_n, \quad 1 \leq k \leq K \qquad n = 0, 1 \ldots (3.3)$$

and (3.1) immediately follows from (2.3) and (2.6) with the help of (Ross 1983) and (Stoyan 1983). Under the foregoing independence assumptions, the inequalities (3.2) are now simple consequences of (3.1) and (2.10)–(2.11) since $\sigma_n^k \leq \sum_{i=1}^k \sigma_n^i$.

□

Since

$$E[\sigma_n^k] = E[u_n^k \cdot \bar{\sigma}_n] = m(B), \quad 1 \leq k \leq K \qquad n = 0, 1 \ldots (3.4)$$

we see that (3.3) cannot hold in the stronger sense of the ordering $\leq_{st}$, for this would otherwise imply $\sigma_n^k =_{st} u_n^k \cdot \bar{\sigma}_n$. Therefore, (3.1)–(3.2) cannot be expected to hold in the ordering $\leq_{st}$ at least in the transient regime.

The scalar inequalities $R_n^k \leq_{icx} U_n + \sum_{i=1}^k \sigma_n^i, 1 \leq k \leq K$, are usually not sufficient to imply a comparison between T_n and S_n. However, for $GI/D/1$ systems, this simple comparison result already yields a complete answer to the question discussed in this paper.

Corollary 3.2. *Assume conditions* **(A1)–(A4)** *to hold with deterministic service times, i.e., for some fixed constant* $\Delta > 0$,

$$\sigma_n^k = \Delta, \quad 1 \leq k \leq K. \qquad n = 0, 1, \dots \tag{3.5}$$

Then, we have the stochastic ordering relations

$$T_n \leq_{icx} S_n. \qquad n = 0, 1 \dots \tag{3.6}$$

Proof. Fix $n = 0, 1, \dots$. For $GI/D/1$ systems, we have $W_n^1 = \dots = W_n^K$ so that $R_n^1 = \dots = R_n^K = W_n^1 + \Delta$ and $T_n = \max_{1 \leq k \leq K} R_n^k = W_n^1 + \Delta$. The result is now immediate from (3.2).

□

Theorem 3.1 also yields the following comparison result for $M/M/1$ systems in statistical equilibrium.

Corollary 3.3. *Assume conditions* **(A1)–(A4)** *to hold with Poisson arrivals and exponential service times, i.e., for some* $\lambda > 0$ *and* $\mu > 0$, *we have*

$$A(t) = 1 - e^{-\lambda t} \quad \text{and} \quad B(t) = 1 - e^{-\mu t}, \quad t \geq 0 \tag{3.7}$$

Under (2.12), the comparison

$$E[T_\infty^{(K)}] \leq E[S_\infty^{(K)}] \qquad K = 1, 2, \dots \tag{3.8}$$

holds.

Proof.

□

IV. A CONCAVITY RESULT FOR THE FORK–JOIN QUEUE

In this section we show that the response time of the n^{th} customer in the Fork–Join queue is increasing and concave in K. Loosely, the concavity comes about because the response time for a job is the largest order statistic of the response times of its constituent tasks. It is a simple matter to show that the expectation of the maximum of a set of

K i.i.d. rvs is increasing and concave in K, and as $\{R_n^1, \ldots, R_n^K\}$ form a set of weakly correlated (in fact associated) random variables [NT,BMSa], their maximum exhibits a very similar behavior. Our principal result in this section is a stronger version of this statement, namely

Theorem 4.1. *For each $n = 0, 1, \ldots$, $\{T_n^{(K)}, K = 1, 2, \ldots\}$ is SICV(st), i.e., for every increasing mapping $f : \mathbb{R}_+ \to \mathbb{R}$, the mapping $K \to E[f(T_n^{(K)})]$ is increasing concave.*

Proof. Fix $n = 0, 1, \ldots$. As pointed out in (Shaked and Shantikumar 1988), it suffices to show that for every $t \geq 0$, the mapping $K \to P[T_n^{(K)} > t]$ is increasing concave. To do so, we first observe that

$$W_n^k = \Phi_n\left(\sigma_m^k,\ 0 \leq m < n; \tau_{m+1},\ 0 \leq m < n\right), \quad 1 \leq k \leq K \tag{4.1}$$

for some mapping $\Phi_n : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$. ¿From this representation, we readily see under the assumptions **(A1)–(A4)** that the rvs $\{R_n^1, \ldots, R_n^K\}$ are conditionally i.i.d. given $\tau_1, \ldots, \tau_n$. Therefore, for every $t \geq 0$, we have

$$\begin{aligned}
P[T_n^{(K)} \leq t] &= P\left[R_n^k \leq t,\ 1 \leq k \leq K\right] \\
&= E\left[P\left[R_n^k \leq t,\ 1 \leq k \leq K | \tau_1, \ldots, \tau_n\right]\right] \\
&= E\left[\prod_{k=1}^{K} P\left[R_n^k \leq t\ | \tau_1, \ldots, \tau_n\right]\right] &(4.2)\\
&= E\left[F_n(t; \tau_1, \ldots, \tau_n)^K\right] &(4.3)
\end{aligned}$$

where we have set

$$F_n(t; t_1, \ldots, t_n) := P\left[\Phi_n\left(\sigma_m^k,\ 0 \leq m < n; t_{m+1},\ 0 \leq m < n\right) + \sigma_n^k \leq t\ \right] \tag{4.4}$$

for every $(t_1, \ldots, t_n)$ in $\mathbb{R}_+^n$; this definition is clearly independent of k owing to **(A3)**. In passing from (4.2) to (4.3) we have used the independence of the service and arrival sequences. It is now plain from (4.3) that the mapping $K \to P[T_n^{(K)} > t]$ is increasing concave.

□

The following corollary is a simple consequence of Theorem 4.1.

Corollary 4.2. *Fix $n = 0, 1, \ldots$. For each $a \geq 0$, the mapping $K \to E\left[(T_n^{(K)} - a)^+\right]$ is increasing concave with*

$$\lim_K \frac{1}{K} E\left[(T_n^{(K)} - a)^+\right] = 0 \tag{4.5}$$

Proof. Fix $n = 0, 1, \ldots$. For each $K = 1, 2, \ldots$, it is known [BMSa] that the rvs $\{R_n^1, \ldots, R_n^K\}$ are associated. Therefore,

$$T_n^{(K)} \leq_{st} \max_{1 \leq k \leq K} \bar{R}_n^k \tag{4.6}$$

where the R_+^K–valued rv $(\bar{R}_n^1, \ldots, \bar{R}_n^K)$ is an independent version of the rv R_n, i.e., the rvs $\{\bar{R}_n^1, \ldots, \bar{R}_n^K\}$ are mutually independent with $\bar{R}_n^k =_{st} R_n^k$, $1 \leq k \leq K$. Consequently, for each $a \geq 0$, we have

$$E\left[(T_n^{(K)} - a)^+\right] \leq E\left[T_n^{(K)}\right] \leq E\left[\max_{1 \leq k \leq K} \bar{R}_n^k\right]. \tag{4.7}$$

Since the non–negative rvs $\{\bar{R}_n^1, \ldots, \bar{R}_n^K\}$ are i.i.d. integrable rvs, a simple application of Proposition C.3 yields

$$\lim_K \frac{1}{K} E\left[\max_{1 \leq k \leq K} \bar{R}_n^k\right] = 0 \tag{4.8}$$

and the convergence (4.5) follows by combining (4.7) and (4.8).

□

V. CONVEXITY RESULTS FOR QUEUES WITH BERNOULLI ROUTING

In this section we show that the response time of the n^{th} customer in a system of K parallel queues with Bernoulli routing is bounded below by a linear increasing function of K, implying that for sufficiently large K, the Fork–Join queue will offer lower response times, both in transient and steady state, than the system of queues with Bernoulli routing. The following result proves useful.

Theorem 5.1. *For each $n = 0, 1, \ldots$, each of the sequences $\{U_n^{(K)},\ K = 1, 2, \ldots\}$ and $\{S_n^{(K)},\ K = 1, 2, \ldots\}$ is increasing in the sense of the ordering $\leq_{icx}$, i.e.,*

$$U_n^{(K)} \leq_{icx} U_n^{(K+1)} \quad \text{and} \quad S_n^{(K)} \leq_{icx} S_n^{(K+1)}. \qquad K = 1, 2, \ldots \tag{5.1}$$

Proof. Fix $K = 1, 2, \ldots$. The $\{0,1\}$–valued rvs $\{\beta_n^{(K)},\ n = 0, 1, \ldots\}$ defined by

$$\beta_n^{(K)} = \delta(\nu_n^{(K)}, 1) \qquad n = 0, 1, \ldots (5.2)$$

form an i.i.d. sequence with

$$P[\beta_n^{(K)} = 1] = \frac{1}{K} = 1 - P[\beta_n^{(K)} = 0]. \qquad n = 0, 1, \ldots (5.3)$$

Moreover, this sequence of Bernoulli rvs $\{\beta_n^{(K)},\ n = 0, 1, \ldots\}$ is independent of the sequence of i.i.d. rvs $\{\bar{\sigma}_n^{(K)},\ n = 0, 1, \ldots\}$ defined by

$$\bar{\sigma}_n^{(K)} = \sum_{k=1}^{K} \sigma_n^k. \qquad n = 0, 1, \ldots (5.4)$$

Now consider the $I\!R_+$–valued rvs $\{Z_n^{(K)},\ n = 0, 1, \ldots\}$ which are generated through the Lindley recursion

$$Z_{n+1}^{(K)} = \left[Z_n^{(K)} + \beta_n^{(K)} \bar{\sigma}_n^{(K)} - \tau_{n+1} \right]^+, \qquad n = 0, 1, \ldots (5.5)$$
$$Z_0^{(K)} = 0.$$

and which are the successive customer waiting times in a $GI/GI/1$ queue with interarrival times $\{\tau_{n+1},\ n = 0, 1, \ldots\}$ and service times $\{\beta_n^{(K)} \bar{\sigma}_n^{(K)},\ n = 0, 1, \ldots\}$. Under the foregoing assumptions, it is plain from (2.10)–(2.11) that

$$U_n^{(K)} =_{st} Z_n^{(K)} \quad \text{and} \quad S_n^{(K)} =_{st} Z_n^{(K)} + \bar{\sigma}_n^K. \qquad n = 0, 1, \ldots (5.6)$$

¿From Claim 2 of Proposition C.1 we conclude that

$$\beta_n^{(K)} \bar{\sigma}_n^{(K)} \leq_{icx} \beta_n^{(K+1)} \bar{\sigma}_n^{(K+1)} \qquad n = 0, 1, \ldots (5.7)$$

so that

$$Z_n^{(K)} \leq_{icx} Z_n^{(K+1)} \qquad n = 0, 1, \ldots (5.8)$$

by making use of ([Ross]). The monotonicity of the sequence $\{U_n^{(K)},\ K=1,2,\ldots\}$ now follows from (5.6). That the sequence $\{S_n^{(K)},\ K=1,2,\ldots\}$ is increasing in the ordering $\leq_{icx}$ then follows from (5.6) and (5.8) upon observing that $\bar{\sigma}_n^{(K)} \leq \bar{\sigma}_n^{(K+1)}$.

□

We are now in the position to prove the following result.

Theorem 5.2. *Fix $n=0,1,\ldots$. For every increasing convex mapping $f:\mathbb{R}_+ \to \mathbb{R}$ with $f(0)=0$, the inequality*

$$E[f(S_n^{(K)})] \geq E[f(S_n^{(1)})] + (K-1)E[f(\sigma_n^1)]. \qquad K=1,2,\ldots \quad (5.9)$$

holds true.

Proof. Fix $n=0,1,\ldots$. From Theorem 5.1

$$\begin{aligned} S_n^{(K)} &=_{st} U_n^{(K)} + \bar{\sigma}_n^{(K)} \\ &\geq_{icx} U_n^{(1)} + \bar{\sigma}_n^{(K)} \end{aligned} \qquad K=1,2,\ldots \quad (5.10)$$

since the rvs $U_n^{(1)}$ and $\sigma_n^{(K)}$ are independent. Consequently, for any increasing convex mapping $f:\mathbb{R}_+ \to \mathbb{R}$ with $f(0)=0$, we conclude from Lemma C.2 that

$$E[f(S_n^{(K)})] \geq E[f(U_n^{(1)} + \sigma_n^{(1)}] + E[f(\sigma_n^2 + \ldots + \sigma_n^K)]. \qquad K=1,2,\ldots \quad (5.11)$$

The inequality (5.9) follows immediately from (5.11) upon applying Lemma C.2 to the second term in the right handside of (5.11) and using the fact that the rvs $\{\sigma_n^2,\ldots,\sigma_n^K\}$ are i.i.d.

□

As an immediate consequence of Theorems 5.1–5.2, we have the following result which is to be compared with Corollary 4.2.

Corollary 5.3. *Fix $n=0,1,\ldots$. For each $a \geq 0$, the mapping $K \to E\left[(S_n^{(K)}-a)^+\right]$ is increasing with*

$$\underline{\lim}_K \ \frac{1}{K} E\left[(S_n^{(K)}-a)^+\right] \geq E\left[(\sigma_n^1 - a)^+\right]. \quad (5.12)$$

As $S_n^{(K)}$ is lower bounded by a function that is increasing and convex in K, and as T_n^K is concave in K, it follows that for sufficiently large K, the response time of the n^{th} job in the Fork Join queue will be lower than that of the n^{th} job in the system of queues with Bernoulli routing. We have been unable to show that $S_n^{(K)}$ is increasing convex. If this could be shown, and as the two systems are identical when $K = 1$, the following stronger result would immediately follow: There is some K_0 such that for all $K < K_0$, $S_n^{(K)} \leq_{icx} T_n^K$ and for all $K \geq K_0$, $T_n^{(K)} \leq_{icx} S_n^K$.

VI. HEAVY TRAFFIC

In section VII we shall compare the Fork–Join queue and the system of parallel queues with Bernoulli routing in their heavy traffic regime. To do so, we need the heavy traffic diffusion limits for each of these two systems, which we now present. Following the approach of Iglehart and Whitt ([IgWhitt]), we consider sequences of stable Fork–Join queues and of stable systems of parallel queues with Bernoulli routing, say indexed by $r \geq 1$, approaching instability as $r \uparrow \infty$. In order to simplify this discussion, we assume that the service times and routing variables remain unchanged and that only the interarrival times vary with r so as to let the utilization increase to the critical value 1.

More precisely, as in Section II, for each $r \geq 1$, we start with the integrable $\mathbb{R}_+$-valued rvs $\{\tau_{n+1}(r),\ n = 0, 1, \ldots\}$, the integrable $\mathbb{R}_+^K$-valued rvs $\{\sigma_n,\ n = 0, 1, \ldots\}$, and the $\{1, \ldots, K\}$-valued rvs $\{\nu_n,\ n = 0, 1, \ldots\}$ under conditions **(A1)–(A4)**. We define the corresponding Fork–Join queue and the system of parallel queues with Bernoulli routing through (2.3)–(2.5) and (2.6)–(2.8), respectively. For all quantities of interest, we explicitly incorporate the dependency on r.

In addition to conditions **(A1)–(A4)**, we enforced the following heavy traffic conditions **(HT1)–(HT2)**, where

(HT1): There exist finite constants $\gamma > 0$ and $\sigma_0 \geq 0$ such that

$$\lim_r \sqrt{r}\,[m(A(r)) - m(B)] = \gamma$$

$$\lim_r \operatorname{var}(A(r)) = \sigma_0^2$$

as $r \uparrow \infty$;

(HT2): There exists $\epsilon > 0$ such that

$$\sup_{r,n} \left\{ E[|\sigma_n^1|^{2+\epsilon}], E[|\tau_{n+1}(r)|^{2+\epsilon}] \right\} < \infty.$$

For our purposes, we find it convenient to realize the heavy traffic conditions **(HT1)–(HT2)** by selecting the interarrival distributions $\{A(r),\ r \geq 1\}$ such that

$$m(A(r)) = m(B) + \frac{\gamma}{\sqrt{r}}, \quad r \geq 1 \tag{6.1}$$

with $\lim_r \operatorname{var}(A(r)) = \sigma_0^2$.

Throughout the discussion, let $\{B_t^0,\ t \geq 0\}$, $\{B_t^1,\ t \geq 0\}$, …, $\{B_t^K,\ t \geq 0\}$ denote $K+1$ independent $I\!R$–valued standard Brownian motions.

VI.1. Results for the Fork–Join queue

In the following proposition, we summarize the relevant results obtained in (Varma and Makowski [REF]) for the Fork–Join queue. The results are stated in terms of the $I\!R^K$–valued process $\{\hat{R}_t,\ t \geq 0\}$ which is defined componentwise by

$$\hat{R}_t^k = \sup_{0 \leq s \leq t} \left\{ \sigma_0 B_s^0 + \sigma(B) B_s^k - \gamma s \right\}, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{6.2}$$

Theorem 6.1. *As $r \uparrow \infty$, we have*

$$\left\{\frac{R_{[rt]}(r)}{\sqrt{r}},\ t \geq 0\right\} \Longrightarrow \{\hat{R}_t,\ t \geq 0\} \tag{6.3a}$$

and

$$\left\{\frac{T_{[rt]}(r)}{\sqrt{r}},\ t \geq 0\right\} \Longrightarrow \{\max_{1 \leq k \leq K} \hat{R}_t^k,\ t \geq 0\}. \tag{6.3b}$$

Moreover, since $\gamma > 0$, the Markov process $\{\hat{R}_t,\ t \geq 0\}$ has a stationary distribution $\hat{R}_\infty = (\hat{R}_\infty^1, \ldots, \hat{R}_\infty^K)$ given by

$$\hat{R}_\infty^k = \sup_{t \geq 0} \left\{ \sigma_0 B_t^0 + \sigma(B) B_t^k - \gamma t \right\}, \quad 1 \leq k \leq K \tag{6.4}$$

and

$$\frac{R_\infty(r)}{\sqrt{r}} \Longrightarrow \hat{R}_\infty. \tag{6.5}$$

It follows from Theorem 6.1 that

$$\frac{T_\infty(r)}{\sqrt{r}} \Longrightarrow \max_{1\le k\le K} \hat{R}^k_\infty. \tag{6.6}$$

Varma and Makowski have shown that the identically distributed rvs $\{\hat{R}^1_\infty, \ldots, \hat{R}^K_\infty\}$ are also associated; it is also well known (Harrison) that each of these rvs is exponentially distributed, i.e., for each $1 \le k \le K$, we have

$$P[\hat{R}^k_\infty > t] = e^{-\alpha t},\ t \ge 0, \quad \text{with} \quad \alpha = \frac{2\gamma}{\sigma_0^2 + \sigma^2(B)}. \tag{6.7}$$

VI.2. Results for the system of parallel queues

The heavy traffic limit for the system of queues with Bernoulli routing is presented next, and follows from standard results available in (Harrison [REF]). To do so, we fix $1 \le k \le K$, and consider the corresponding single–server queue embedded in the system of queues with Bernoulli routing. Elementary arguments show that

$$m(u^k_n \bar{\sigma}_n) = m(B) \qquad n = 0, 1, \ldots \tag{6.8}$$

and

$$\text{var}(u^k_n \bar{\sigma}_n) = \text{var}(B) + (K-1)m(B)^2. \qquad n = 0, 1, \ldots \tag{6.9}$$

Consequently, under **(HT1)–(HT2)**, the k^{th} single–server queue (and therefore, the entire system of parallel queues with Bernoulli routing) reaches instability according to

$$\lim_r \sqrt{r}[m(A(r)) - m(u^k_n \bar{\sigma}_n)] = \gamma \tag{6.10}$$

and (6.).

We can now invoke classical results from the heavy traffic theory for the $GI/GI/1$ queue. To do so, we define the $\mathbb{R}$–valued process $\{\hat{S}_t,\ t \ge 0\}$ by

$$\hat{S}_t = \sup_{0\le s\le t} \left\{\sigma_0 B^0_s + \sigma(K) B^1_s - \gamma s\right\}, \quad t \ge 0 \tag{6.11}$$

where we have set

$$\sigma^2(K) = \sigma^2(B) + (K-1)m(B)^2 = m(B)^2\left[C^2(B) + (K-1)\right]. \tag{6.12}$$

Making use of (2.10)–(2.11), we readily conclude to the followingresult.

Theorem 6.1. *As $r \uparrow \infty$, we have*

$$\{\frac{S_{[rt]}(r)}{\sqrt{r}},\ t \geq 0\} \Longrightarrow \{\hat{S}_t,\ t \geq 0\}. \tag{6.13}$$

Moreover, since $\gamma > 0$, the Markov process $\{\hat{S}_t,\ t \geq 0\}$ has a stationary distribution $\hat{S}_\infty$ given by

$$\hat{S}_\infty = \sup_{t\geq 0}\left\{\sigma_0 B_t^0 + \sigma(K)B_t^1 - \gamma t\right\} \tag{6.14}$$

and

$$\frac{S_\infty(r)}{\sqrt{r}} \Longrightarrow \hat{S}_\infty. \tag{6.15}$$

Here we have

$$P[\hat{S}_\infty > t] = e^{-\alpha(K)t}, /t \geq 0, \quad \text{with} \quad \alpha(K) = \frac{2\gamma}{\sigma_0^2 + \sigma(K)^2}. \tag{6.16}$$

VII. COMPARISONS IN HEAVY TRAFFIC

Combining the heavy traffic results of Section VI, we observe for every $a \geq 0$ the relations

$$\lim_r \frac{E\left[(T_\infty(r)-a)^+\right]}{E\left[(S_\infty(r)-a)^+\right]} = \lim_r \frac{\frac{1}{\sqrt{r}}E\left[(T_\infty(r)-a)^+\right]}{\frac{1}{\sqrt{r}}E\left[(S_\infty(r)-a)^+\right]} = \frac{E[\hat{T}_\infty]}{E[\hat{S}_\infty]}. \tag{7.1}$$

The relation (7.1) can be exploited as follows. If $E[\hat{T}_\infty] < E[\hat{S}_\infty]$, then (7.1) implies that for large r, say for $r \geq r(a)$, $E\left[(T_\infty(r)-a)^+\right] < E\left[(S_\infty(r)-a)^+\right]$. Now, if we could select $r(a)$ uniformly in a, then we would conclude that for high utilization, the response time of the Fork–Join queue with K processors is smaller than the response time of the corresponding system of parallel queues with Bernoulli routing, the comparison being in the sense of the ordering $\leq_{icx}$.

To carry out the program outlined above, we need to evaluate the right handside of (7.1) as we now do. ¿From (6.16) we conclude that

$$E[\hat{S}_\infty] = \frac{1}{\alpha(K)} = \frac{\sigma_0^2 + \sigma(K)^2}{2\gamma}. \tag{7.2}$$

As pointed out in (Varma and Makowski [REF]), the explicit evaluation of $E[\hat{T}_\infty]$ is difficult, if not impossible, except in very few cases. However, the stochastic bounds derived in (Baccelli, Makowski and Shwartz [REF]) still hold in heavy traffic as shown by Varma and Makowski ([REF]). In particular, we have the bounds

$$\frac{\sigma^2(B)}{2\gamma} H_K \leq E[\hat{S}_\infty] \leq \frac{\sigma_0^2 + \sigma^2(B)}{2\gamma} H_K. \tag{7.3}$$

The first inequality in (7.3) is the heavy traffic equivalent to the lower bound obtained by Baccelli, Makowski and Shwartz (REF). This lower bound holds in the ordering $\leq_{icx}$ and corresponds to the response time of a Fork–Join queue with deterministic arrivals, everything else being equal. In heavy traffic this is characterized by $\sigma_0^2 = 0$, in which case the rvs $\{\hat{R}^1_\infty, \ldots, \hat{R}^K_\infty\}$ are i.i.d. The second inequality in (7.3) is the heavy traffic version of the upper bound by association developed in (Baccelli, Makowski and Shwartz [REF], Nelson and Tantawi [REF]). This upper bound is in the ordering $\leq_{st}$ and can be interpreted as the response time of a system of parallel queues with independent arrivals and Join synchronization.

¿From (7.1)–(7.3), we conclude that

$$\frac{\sigma^2(B)}{\sigma_0^2 + \sigma^2(K)} H_K \leq \frac{E[\hat{T}_\infty]}{E[\hat{S}_\infty]} \leq \frac{\sigma_0^2 + \sigma^2(B)}{\sigma_0^2 + \sigma^2(K)} H_K, \tag{7.4}$$

or equivalently,

$$L_K \leq \frac{E[\hat{T}_\infty]}{E[\hat{S}_\infty]} \leq U_K, \tag{7.5}$$

where for convenience we have defined

$$L_K := \frac{\sigma^2(B)}{\sigma_0^2 + \sigma^2(K)} H_K \quad \text{and} \quad U_K := \frac{\sigma_0^2 + \sigma^2(B)}{\sigma_0^2 + \sigma^2(K)} H_K. \tag{7.6}$$

We first discuss the implications of (7.4) in a general $GI/GI/1$ setting, and then consider several special situations.

Theorem 7.1. *Assume the heavy traffic conditions* **(HT1)–(HT2)** *to be enforced with the additional constraint*

$$\sigma_0^2 + \sigma^2(B) \leq 2m(B)^2. \tag{7.7}$$

Then, for all $K = 1, 2, \ldots$, we have

$$E[\hat{T}_\infty] \leq E[\hat{S}_\infty]. \tag{7.8}$$

Proof. We shall readily obtain (7.8) from (7.5) provided it can be shown that $U_K \leq 1$ or equivalently, that

$$(\sigma_0^2 + \sigma^2(B))H_K \leq \sigma_0^2 + \sigma^2(B) + (K-1)m(B)^2. \tag{7.9}$$

We now show that (7.9) holds for all $K = 1, 2, \ldots$ by induction. For $K = 1$, the inequality (7.9) trivially holds as an equality, thus establishing the basis step. To proceed, we assume that (7.9) holds for some $K \geq 1$, in which case

$$\begin{aligned}(\sigma_0^2 + \sigma^2(B))H_{K+1} &\leq \sigma_0^2 + \sigma^2(B) + (K-1)m(B)^2 + \frac{\sigma_0^2 + \sigma^2(B)}{K+1} \\ &\leq \sigma_0^2 + \sigma^2(B) + Km(B)^2\end{aligned}$$

where we have used (7.7) to obtain the second inequality. This completes the inductive step. □

We now specialize Theorem 7.1. to the $M/GI/1$ case. We have $m(A(r)) = \frac{1}{\lambda(r)}$ and $\text{var}(A(r)) = \frac{1}{\lambda(r)^2}$ for all $r \geq 1$, where $\lambda(r)$ is chosen according to (6.1), so that

$$\lambda(r) = \frac{\sqrt{r}}{\gamma + \sqrt{r}m(B)}, \quad r \geq 1. \tag{7.11}$$

Under **(HT1)–(HT2)**, we get $\lim_r \lambda(r) = m(B)^{-1}$, whence $\sigma_0^2 = m(B)^2$, and the bounds (7.5) take the form

$$L_K = \frac{C^2(B)}{K + C^2(B)} \cdot H_K \quad \text{and} \quad U_K = \frac{1 + C^2(B)}{K + C^2(B)} \cdot H_K. \tag{7.12}$$

Since $\lim_K U_K = 0$, we see that for large enough K, in fact for $K < K(B)$, with $K(B) := \inf\{K \geq 1 : U_K \leq 1\}$, the heavy traffic performance of the Fork–Join queue with K processors is better than the heavy traffic performance of the corresponding system of parallel queues with Bernoulli routing. However, we see from (7.5) and (7.12) that the comparison for a small number of processors is crucially dependent on the variability of the service time distribution B through its coefficient of variation $C^2(B)$, and consequently, for any given number $K \geq 2$ of processors, there are service time distributions for which $S_\infty \leq T_\infty$, and others for which $T_\infty \leq S_\infty$.

Corollary 7.2. *Consider $M/GI/1$ type systems, i.e., Poisson arrivals, and assume the heavy traffic conditions* **(HT1)–(HT2)** *with $C^2(B) \leq 1$. Then, for all $K = 1, 2, \ldots$, we also have (7.8).*

Proof. As pointed earlier, we have $\sigma_0^2 = m(B)^2$, which implies that (7.7) is equivalent to $C^2(B) \leq 1$, and the result then follows from Theorem 7.1.

□

Observe that this last result applies to the $M/M/1$ situation where $C^2(B) = 1$. In fact, we find from (7.12) that

$$L_K = \frac{H_K}{K+1} \quad \text{and} \quad U_K = \frac{2H_K}{K+1}, \tag{7.13}$$

and a direct inspection shows that $U_K \leq 1$, whence

$$(M/M/1): \quad E[\hat{T}_\infty] \leq E[\hat{S}_\infty]. \tag{7.14}$$

The $GI/M/1$ situation is characterized by the relation $\sigma^2(B) = m(B)^2$, so that

$$L_K := \frac{m(B)^2}{\sigma_0^2 + Km(B)^2} H_K \quad \text{and} \quad U_K := \frac{\sigma_0^2 + m(B)^2}{\sigma_0^2 + Km(B)^2} H_K. \tag{7.15}$$

Theorem 7.1 easily specializes to this case and reads

Corollary 7.3. *Consider $GI/M/1$ type systems, i.e., exponential services, and assume the heavy traffic conditions* **(HT1)–(HT2)** *with*

$$\sigma_0^2 \leq m(B)^2. \tag{7.16}$$

Then, for all $K = 1, 2, \ldots$, *we also have (7.8).*

VIII. LIGHT TRAFFIC

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APPENDIX A
STOCHASTIC ORDERING

APPENDIX B
STOCHASTIC CONVEXITY

In this appendix, we briefly recall several notions of stochastic convexity which have been recently introduced by Shaked and Shantikumar [11,12]: Throughout, Θ is a convex subset of $\mathbb{R}$ and $\{X(\theta),\ \theta \in \Theta\}$ is a collection of $\mathbb{R}$–valued rvs. For any Borel mapping $f : \mathbb{R} \to \mathbb{R}$, we define the mapping $\hat{f} : \Theta \to \mathbb{R}$ by

$$\hat{f}(\theta) := E[f(X(\theta))], \quad \theta \in \Theta \tag{2.1}$$

whenever these expectations exist. The collection of rvs $\{X(\theta),\ \theta \in \Theta\}$ is then said to be

1. stochastically increasing (resp. decreasing) convex in the usual stochastic ordering – in short SICX(st) (resp. SDCX(st)) – $\hat{f}$ is increasing (resp. decreasing) convex whenever f is increasing;

2. stochastically increasing (resp. decreasing) convex – in short SICX (resp. SDCX) – if $\hat{f}$ is increasing (resp. decreasing) convex whenever f is increasing convex;

3. stochastically increasing convex in the sample path sense – in short SICX(sp) – if for any four points θ_i, $i = 1, \ldots, 4$, in Θ, such that $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3$, there exist four rvs $\tilde{X}_i$, $i = 1, \ldots, 4$, defined on a common probability space such that $\tilde{X}_i =_{st} X(\theta_i)$, $i = 1, \ldots, 4$, and

$$\tilde{X}_j \le \tilde{X}_4, \quad j = 1, 2, 3 \quad \text{and} \quad \tilde{X}_2 + \tilde{X}_3 \le \tilde{X}_1 + \tilde{X}_4 \quad a.s. \tag{2.2}$$

A few words on these definitions: When the rvs $\{X(\theta),\ \theta \in \Theta\}$ are non–negative rvs, we note in the definition of SICX(st) and SICX that we need only consider $\mathbb{R}_+$–valued mappings f, in which case $\hat{f}$ is always well defined. Moreover, when Θ is a subset of $\{0, 1, \ldots\}$, convexity is understood as integer convexity.

The following implications were discussed in (Shaked and Shantikumar [11,12]):

$$\text{SICX(st)} \Longrightarrow \text{SICX(sp)} \Longrightarrow \text{SICX}. \tag{2.3}$$

In general, the implications SICX(sp) $\Longrightarrow$ SICX(st) and SICX $\Longrightarrow$ SICX(sp) are not true as can be seen on simple counterexamples.

APPENDIX C

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Let $\{X_n,\ n = 1, 2, \ldots\}$ be a sequence of $\mathbb{R}_+$–valued rvs. As usual we define the partial sums $\{S_n,\ n = 1, 2, \ldots\}$ by

$$S_n := \sum_{k=1}^{n} X_k \qquad n = 1, 2, \ldots (C.1)$$

with the convention $S_0 = 0$. Moreover, let $\{b_n,\ n = 1, 2, \ldots\}$ be a sequence of $\{0, 1\}$–valued rvs such that

$$P(b_n = 1) = \frac{1}{n} = 1 - P(b_n = 0). \qquad n = 1, 2, \ldots (C.2)$$

Proposition C.1. *Assume the sequences $\{X_n,\ n = 1, 2, \ldots\}$ and $\{b_n,\ n = 1, 2, \ldots\}$ to be mutually independent. If the non–negative rvs $\{X_n,\ n = 1, 2, \ldots\}$ are i.i.d., then the following statements hold true.*

1. *The stochastic ordering relations*

$$X_n \leq_{icx} b_n S_n \qquad n = 1, 2, \ldots (C.3)$$

hold true; and

2. *The sequence $\{b_n S_n,\ n = 1, 2, \ldots\}$ is increasing in the sense of the ordering $\leq_{icx}$, i.e.,*

$$b_n S_n \leq_{icx} b_{n+1} S_{n+1}. \qquad n = 1, 2, \ldots (C.4)$$

The proof of Proposition A.1 makes use of a well–known property of increasing convex funtions which we state here for easy reference.

Lemma C.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be any increasing convex mapping such that $f(0) = 0$. The inequality*

$$\sum_{k=1}^{n} f(x_k) \leq f(\sum_{k=1}^{n} x_k) \qquad n = 1, 2, \ldots (C.5)$$

holds true for arbitrary $x_k \geq 0$, $1 \leq k \leq n$.

Proof of Proposition A.1. First we observe for every mapping $f : I\!R_+ \to I\!R_+$ that

$$E[f(b_n S_n)] = \frac{1}{n} E[f(S_n)] + (1 - \frac{1}{n}) f(0). \qquad n = 1, 2, \ldots (C.6)$$

In order to establish Claim 1, we need only show that for every increasing convex mapping $f : I\!R_+ \to I\!R_+$, the inequality

$$E(f(X_n)) \leq \frac{1}{n} E(f(S_n)) + (1 - \frac{1}{n}) f(0). \qquad n = 1, 2, \ldots (A.7)$$

holds true. Of course, there is no loss of generality in assuming $f(0) = 0$, in which case (A.7) reduces to

$$E(f(X_n)) \leq \frac{1}{n} E(f(S_n)) \qquad n = 1, 2, \ldots (A.8)$$

This last inequality is an immediate consequence of Lemma A.2 upon observing that

$$nE(f(X_n)) = E(\sum_{k=1}^{n} f(X_k)). \qquad n = 1, 2, \ldots (A.9)$$

since the non-negative rvs $\{X_n,\ n = 1, 2, \ldots\}$ are i.i.d.

Claim 2 is equivalent to the fact that for every increasing convex mapping $f : I\!R_+ \to I\!R_+$, the mapping $n \to E(f(b_n S_n))$ is increasing. Again there is no loss of generality in assuming $f(0) = 0$, as we do from now on, in which case we only need to show that

$$\frac{1}{n} E(f(S_n)) \leq \frac{1}{n+1} E(f(S_{n+1})). \qquad n = 1, 2, \ldots (A.10)$$

This is now done by induction on n.

- The basis step: For $n = 1$, (A.10) reduces to the inequality

$$E(f(X_1)) \leq \frac{1}{2} E(f(S_2)) \qquad (A.11)$$

which is exactly (A.8) (with $n = 2$ since $X_1 =_{st} X_2$).

• The induction step: Assuming now that (A.10) indeed holds for some $n = m \geq 1$, we want to show that (A.10) also holds for $n = m + 1$. With this in mind, we define the mapping $f_m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$f_m(x) := E(f(S_{m-1} + x)), \quad x \geq 0 \tag{A.12}$$

and observe that f_m is increasing convex whenever f is increasing convex.

Under the enforced asssumptions, the rv S_{m-1} is independent of the rvs $X_m + X_{m+1}$, so that

$$\begin{aligned} E(f(S_{m+1})) &= E(f_m(X_m + X_{m+1})) \\ &= f_m(0) + E(f_m(X_m + X_{m+1}) - f_m(0)). \end{aligned} \tag{A.13}$$

Since (A.10) holds for $n = 2$ by virtue of the basis step, we conclude that

$$\begin{aligned} E(f(S_{m+1})) &\geq f_m(0) + 2E(f_m(X_{m+1}) - f_m(0)) \\ &= 2E(f_m(X_m)) - f_m(0) \\ &= 2E(f(S_m)) - E(f(S_{m-1})). \end{aligned} \tag{A.14}$$

where the first equality follows from the fact that $X_m =_{st} X_{m+1}$. Therefore, (A.10) will hold for $n = m + 1$ provided we can show that

$$\frac{1}{m} E(f(S_m)) \leq \frac{1}{m+1} \left(2E(f(S_m)) - E(f(S_{m-1})) \right) \tag{A.15}$$

By simple arithmetic we see that (A.15) is equivalent to (A.10) for $n = m$ and this establishes the induction step.

□

Proposition A.3. *If the i.i.d. rvs* $\{X_n,\ n = 1, 2, \ldots\}$ *are integrable, then*

$$\lim_n \frac{1}{n} E\left(\max_{1 \leq i \leq n} X_i\right) = 0. \tag{A.11}$$

Proof.

□