HEAVY TRAFFIC LIMITS FOR FORK–JOIN QUEUES

by

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ABSTRACT

such as

Fork–join queueing models arise in many application areas, including parallel processing and flexible manufacturing systems. These queues typically exhibit non–product form behavior and their analysis has proven to be difficult. In this work, our objective is to obtain heavy traffic approximations for Fork–Join systems. We first demonstrate the convergence to a diffusion process for appropriately scaled and normalized versions of the response time sequence. Thereby the problem of obtaining the heavy traffic limit for the response time is reduced to the problem of obtaining stationary distribution of this diffusion. This stationary distribution satisfies a second order partial differential equation (PDE) with oblique derivative boundary conditions. Upper and lower bounds for the stationary distribution are obtained using stochastic ordering theory. For the special case of two dimensional Fork–Join queues, we solve the PDE for the stationary distribution to obtain formulae for the heavy traffic limit for all moments of the response time.

1) Use notation $\hat{W}^k_t(\gamma) = \frac{1}{\sqrt{\gamma}} W^k_{[\gamma t]}(\gamma)$

2) ~~Make~~ Combine Lemmas together:

3) change bound by association argument.

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1. Introduction

Fork–join systems arise as models for parallel processing systems. For example, consider a parallel processing system with K processors which is subject to an arrival stream of jobs consisting of multiple tasks (Fig 1). Suppose that the K tasks are independent of each other, so that they may be executed in parallel. If we assume that the number of processors is equal to the number of tasks, all the jobs have the same task graph and all inter–processor communication delays can be ignored, then the system can be modeled by a Fork–Join queue (Fig 2). The problem of interest is to obtain the statistics of the response time of a customer in this system. In general this is a very difficult problem, because the presence of synchronization delays (in addition to queueing delays) destroys the product form property which characterizes Jackson networks.

A number of authors have analysed the Fork–Join queue and one of the first such analysis was carried out by Flatto and Hahn [7]. They considered the case when there were only two queues subject to Poisson arrivals and exponential service times, and were interested in obtaining information about the transform of the steady state queue length statistics of the system. Using advanced tools from complex analysis they were able to solve this problem, h However it was a difficult task to obtain explicit formulae from their final solution, and moreover their analysis was not generalizable to more than two queues. Later Baccelli [1] gave a more general solution to the two queue problem, in the sense that he removed the restriction of exponential services, so that his analysis was applicable to fork-join queues with ~~applied for~~ Poisson arrivals and general service distributions. Nelson and Tantawi [22] and Baccelli, Makowski and Shwartz [2] gave bounds for the average response time for Fork–Join systems made up of K queues, with general inter–arrival and service distributions. They showed that the average response time was lower bounded by a related system with deterministic arrivals, and upper bounded by a system with independent arrivals. Since in either case, the queues become uncorrelated with one another, their response time statistics can be explicitly characterized. Nelson and Tantawi [22] also obtained an approximation for the average response time of a homogeneous system with Poisson arrivals and exponential services. This approximation was obtained by a clever argument that involved both theoretical as well as empirical ~~experimental~~ considerations.

— Agarwala's paper.

In this paper our objective is to carry out an heavy traffic analysis of the Fork–Join queue. We first demonstrate the convergence to a diffusion process for the various queueing delay processes in the queue. The partial differential equation that characterizes the stationary distribution of the diffusion is then given. This equation is solved for the special case of two queues, and with help of the solution we obtain formulae for the heavy traffic limit of the n^{th} moment of the response time.

The heavy traffic results in this paper are combined with light traffic limits to obtain interpolation approximations in [29]. In [30] we present approximations for acyclic Fork–Join networks that are used to model parallel processing systems with precedence constraints among the various tasks in a job. In [32] we present approximations for synchronized queueing networks which are used to model parallel processing systems with precedence constraints among the tasks as well as communication delays between processors.

2. The model.

Consider a system of K single server queues which operates in parallel ~~on an incoming stream of jobs as we now describe~~: Each single server queue has an infinite capacity buffer and operates according to the first–come first–serve discipline. Each job is constituted of K distinct tasks, with the understanding the k^{th} task is to be processed by the k^{th} server, $1 \leq k \leq K$. Therefore, upon arriving into the system, a job splits into its constituting tasks so that the k^{th} task enters the buffer attending the k^{th} server. This is known as the fork synchronization constraint. After a task receives service, it is put in another buffer, the so–called synchronization buffer, until that moment when all other tasks from the same job have finished their service at the other servers. Only when all the tasks that make up a job have been processed, is the job declared serviced and its K tasks all leave simultaneously. This is known as the join synchronization constraint.

The following RVs are defined on a common probability space $(\Omega, \mathcal{F}, P)$. For $n = 0, 1 \ldots$ and $1 \leq k \leq K$, we set

u_{n+1} : Inter–arrival time between the $(n+1)^{rst}$ and n^{th} job;

v_n^k : Service time of the type k^{th} task from the n^{th} job;

W_n^k : Waiting time of the type k^{th} task from the n^{th} job;

R_n^k : Response time of the type k^{th} task from the n^{th} job; and

T_n : System response time of the n^{th} job.

We shall assume that

(Ia): The sequences $\{u_{n+1},\ n = 0, 1, \ldots\}$ and $\{v_n^k,\ n = 0, 1, \ldots\}$, $1 \leq k \leq K$, are mutually independent. Moreover, each one of these sequences is composed of i.i.d. RVs with finite second moments.

For $n = 0, 1 \ldots$, we set

$$u = E[u_{n+1}] < \infty, \quad \sigma_0^2 = \mathrm{Var}[u_{n+1}] < \infty$$

and

$$v^k = E[v_n^k] < \infty, \quad \sigma_k^2 = \mathrm{Var}[v_n^k] < \infty, quad1 \leq k \leq K.$$

Under assumptions **(Ia)**, each queue in the Fork–Join system operates like a $GI/GI/1$ queue. Therefore, upon assuming that the 0^{th} job arrives into an empty system at time $t = 0$, we can write down the Lindley recursion for the sequence of waiting times in the k^{th} queue, i.e., for each $1 \leq k \leq K$,

$$\begin{aligned} W_0^k &= 0 \\ W_{n+1}^k &= [W_n^k + v_n^k - u_{n+1}]^+. \end{aligned} \qquad n = 0, 1 \ldots (2.1)$$

The response time $R_n^k, 1 \leq k \leq K$, is given by

$$R_n^k = W_n^k + v_n^k, \qquad n = 0, 1 \ldots (2.2)$$

and the system response time T_n of the n^{th} job is then given by

$$T_n = \max_{1 \leq k \leq K} R_n^k. \qquad n = 0, 1 \ldots (2.3)$$

We consider the system to be stable if the sequence of queueing delay vectors $\{(W_n^1, \ldots, W_n^K),\ n = 0, 1, \ldots\}$ converges in distribution as $n \uparrow \infty$ to a proper random vector $(W^1, \ldots, W^K)$. It was shown in [2] that system is satble *iff* the condition

$$v^k < u, \quad 1 \leq k \leq K \tag{2.4}$$

holds.

3. Existence of a diffusion limit

In this section, we focus on the task of obtaining heavy traffic diffusion limits for the delay processes in the Fork–Join queue. Following the approach of Iglehart and Whitt [15], we consider a sequence of stable Fork–Join systems, say indexed by $r = 1, 2, \ldots$, approaching instability as $r \uparrow \infty$. We then show that a rescaled K–dimensional stochastic process generated by the vector delay sequence converges weakly to a K–dimensional correlated diffusion process in the non–negative orthant, with normal reflections at the boundaries.

Throughout, all continuous–time processes have sample paths in $D[0,\infty)$, the space of right continuous functions possessing left limits [21]. As usual, $\Rightarrow$ denotes weak convergence. The convergence results give convergence over the interval $[0,\infty)$. However in the proofs we limit ourselves to proving convergence over any finite interval $[0,T]$, since the two cases are equivalent as long as the limiting process obtained has continuous sample paths [21, Thm. 3′, pp. 120].

For each $r = 1, 2 \ldots$, we consider a Fork–Join queues generated by the sequences $\{u_{n+1}(r),\ n = 0, 1, \ldots\}$ and $\{v_n^k(r),\ n = 0, 1, \ldots\}$, $1 \le k \le K$, under assumption **(Ia)**. In addition, we assume **(Ib)**–**(Ic)**, where

(Ib): As $r \uparrow \infty$,

$$\sigma_k(r) \to \sigma_k, \quad 0 \le k \le K,$$
$$[u(r) - v^k(r)]\sqrt{r} \to c_k, \quad 1 \le k \le K.$$

(Ic): For some $\epsilon > 0$,

$$\sup_{r,k} \{E\{\mid u_1(r) \mid^{2+\epsilon}\}, E\{\mid v_1^k(r) \mid^{2+\epsilon}\}\} < \infty.$$

For $r = 1, 2 \ldots$, we define the following partial sums

$$V_0^k(r) = 0,$$
$$V_n^k(r) = v_0^k(r) + \ldots + v_{n-1}^k(r), \quad 1 \le k \le K, \qquad n = 1, 2 \ldots (3.1a)$$

and

$$U_0(r) = 0,$$
$$U_n(r) = u_1(r) + \ldots + u_n(r). \qquad n = 1, 2 \ldots (3.1b)$$

The stochastic processes $\xi^k(r) \equiv \{\xi_t^k(r), t \geq 0\}$, $0 \leq k \leq K$, are then defined by

$$\xi_t^0(r) = \frac{U_{[rt]}(r) - u(r)[rt]}{\sqrt{r}}, \quad t \geq 0 \tag{3.2a}$$

and

$$\xi_t^k(r) = \frac{V_{[rt]}^k(r) - v^k(r)[rt]}{\sqrt{r}}, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{3.2b}$$

Lemma 3.1 *Let $\xi^k \equiv \{\xi_t^k, t \geq 0\}$, $0 \leq k \leq K$, be $K+1$ independent standard Brownian motions. As $r \uparrow \infty$,*

$$(\xi^0(r), \xi^1(r), \ldots, \xi^K(r)) \Rightarrow (\sigma_0 \xi^0, \sigma_1 \xi^1, \ldots, \sigma_K \xi^K) \tag{3.3}$$

in $D[0, \infty)^{K+1}$.

Proof. Equation (2.6) follows directly by Prohorov's functional central limit theorem for triangular arrays [24] under assumptions **(Ia)**–**(Ic)**. ∎

For $r = 1, 2 \ldots$, we set

$$S_0^k(r) = 0$$
$$S_n^k(r) = V_n^k(r) - U_n(r), \ 1 \leq k \leq K \qquad n = 1, 2 \ldots (3.4)$$

and define the stochastic processes $\zeta^k(r) \equiv \{\zeta_t^k(r), t \geq 0\}$, $1 \leq k \leq K$, by

$$\zeta_t^k(r) = \frac{S_{[rt]}^k(r)}{\sqrt{r}}, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{3.5}$$

We also define the stochastic processes $\zeta^k \equiv \{\zeta_t^k, t \geq 0\}, 1 \leq k \leq K$, by

$$\zeta_t^k = \sigma_k \xi_t^k - \sigma_0 \xi_t^0 - c_k t, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{3.6}$$

The process $(\zeta^1, \ldots, \zeta^K)$ is a K-dimensional diffusion process with drift vector c and covariance matrix R given by

$$c = (-c_1, \ldots, -c_K) \tag{3.7}$$

and

$$R = \begin{pmatrix} \sigma_1^2 + \sigma_0^2 & \sigma_0^2 & \ldots & \sigma_0^2 \\ \sigma_0^2 & \sigma_2^2 + \sigma_0^2 & \ldots & \sigma_0^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_0^2 & \sigma_0^2 & \ldots & \sigma_K^2 + \sigma_0^2 \end{pmatrix}. \tag{3.8}$$

Lemma 3.2 shows that the stochastic processes (3.5) generated by the random walk (3.4) converge to $(\zeta^1, \ldots, \zeta^K)$ in the limit. The cross-correlation terms in the matrix R reflect the correlation between the K queues due to the common arrival process.

Lemma 3.2. *As $r \uparrow \infty$,*

$$(\zeta^1(r), \ldots, \zeta^K(r)) \Rightarrow (\zeta^1, \ldots, \zeta^K) \tag{3.9}$$

in $D[0, \infty)^K$.

Proof. Fix $r \geq 1$ and $t \geq 0$. We see from (3.2), (3.4) and (3.5) that

$$\begin{aligned} \zeta_t^k(r) &= \frac{V_{[rt]}^k(r) - U_{[rt]}}{\sqrt{r}} \\ &= \xi_t^k(r) - \xi_t^0(r) - \frac{[rt]}{r}[u(r) - v^k(r)]\sqrt{r}, \quad 1 \leq k \leq K \end{aligned}$$

Assumption **(Ib)** readily implies that as $r \uparrow \infty$,

$$\frac{[rt]}{r}[u(r) - v^k(r)]\sqrt{r} \to c_k t, \quad 1 \leq k \leq K$$

and we conclude to (3.9) by invoking Lemma 3.1 and the Continuous Mapping Theorem [6, Theorem 5.1]. ∎

The Lindley recursion (2.1) for the queueing delays can be reformulated in the following way, which proves very useful in establishing limit theorems. For $r = 1, 2 \ldots$ and $1 \leq k \leq K$, we observe that

$$\begin{aligned} W_n^k(r) &= \max\{S_n^k(r) - S_i^k(r),\ i = 0, 1 \ldots, n\} \\ &= S_n^k(r) - \min\{S_i^k(r),\ i = 0, 1 \ldots, n\}. \qquad n = 0, 1 \ldots \end{aligned} \tag{3.10}$$

Next, we define the stochastic processes $\mu^k(r) \equiv \{\mu_t^k(r), t \geq 0\}$, $1 \leq k \leq K$, by

$$\mu_t^k(r) = \frac{W_{[rt]}^k(r)}{\sqrt{r}}, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{3.11}$$

We also define the stochastic processes $\eta^k \equiv \{\eta_t^k, t \geq 0\}$, $1 \leq k \leq K$, by

$$\eta_t^k = g(\zeta^k)_t, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{3.12}$$

where $g : D[0,\infty) \to D[0,\infty)$ is the the reflection mapping defined by

$$g(x)_t = x_t - \inf_{0 \leq s \leq t} x_s, \quad t \geq 0. \tag{3.13}$$

In Lemma 3.3 we show that the vector process associated with (3.11) converges weakly to a K–dimensional diffusion process (3.12) with drift (3.7) and covariance (3.8). This limiting diffusion stays in the non–negative orthant in $I\!R^K$ and exhibits normal reflections at the boundaries.

Lemma 3.3 *As $r \uparrow \infty$,*

$$(\mu^1(r), \ldots, \mu^K(r)) \Rightarrow (\eta^1, \ldots, \eta^K) \tag{3.14}$$

in $D[0,\infty)^K$.

Proof. From (3.10) and (3.11), we conclude that

$$\mu^k(r) = g(\zeta^k(r)), \quad 1 \leq k \leq K. \qquad r = 1, 2, \ldots \tag{3.15}$$

Since g is a continuous mapping [36], the result follows by the Continuous Mapping Theorem and Lemma 3.2. ∎

For $r = 1, 2 \ldots$, we define the stochastic processes $\eta^k(r) \equiv \{\eta_t^k(r), t \geq 0\}$, $1 \leq k \leq K$, by

$$\eta_t^k(r) = \frac{R_{[rt]}^k(r)}{\sqrt{r}}, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{3.16}$$

The vector process defined by (3.16) converges weakly to the same limiting process as the vector process generated by the waiting times.

Theorem 3.4 *As* $r \uparrow \infty$,

$$(\eta^1(r), \ldots, \eta^K(r)) \Rightarrow (\eta^1, \ldots, \eta^K) \tag{3.17}$$

in $D[0,\infty)^K$.

Proof. Fix $T > 0$. For each $r = 1, 2\ldots$, we see from (2.2) that

$$R_n^k(r) - W_n^k(r) = v_n^k(r), \quad 1 \leq k \leq K \qquad\qquad n = 0, 1\ldots$$

whence

$$\sup_{0 \leq t \leq T} \max_{1 \leq k \leq K} \mid \eta_t^k(r) - \mu_t^k(r) \mid = \frac{1}{\sqrt{r}} \max_{1 \leq k \leq K} \max\{v_n^k(r) : 0 \leq n \leq r\} \xrightarrow{\mathbb{P}} 0$$

as $r \uparrow \infty$ [14]. We conclude (3.17) from Lemma 3.3 and from the Converging Together Theorem [6, Theorem 4.1]. ∎

For $r = 1, 2\ldots$, we define the stochastic processes $\kappa(r) \equiv \{\kappa_t(r), t \geq 0\}$ by

$$\kappa_t(r) = \frac{T_{[rt]}(r)}{\sqrt{r}}, \quad t \geq 0. \tag{3.18}$$

Also define the stochastic process $\kappa \equiv \{\kappa_t, t \geq 0\}$ by

$$\kappa_t = \max_{1 \leq k \leq K} \eta_t^k, \quad t \geq 0. \tag{3.19}$$

The stochastic process (3.18) generated by the end-to-end delays converges weakly to the process (3.19), which is the maximum of K correlated Wiener processes with drift, in the non-negative orthant and normal reflection at the boundaries.

Lemma 3.5 *As* $r \uparrow \infty$,

$$\kappa(r) \Rightarrow \kappa \tag{3.20}$$

in $D[0,\infty)$.

Proof. From (2.3) and (3.17) we conclude that

$$\kappa_t(r) = \max_{1\leq k\leq K} \eta_t^k(r), \quad t \geq 0. \qquad r = 1, 2, \ldots \quad (3.21)$$

The convergence (3.20) now follows from Theorem 3.5 and the Continuous Mapping Theorem, Upon noting that $x \to \max_{1\leq k\leq K} x_k$ is a continuous function on $I\!R^K$. ∎

4. Properties of the diffusion limit

4.1. A Markov property

To present the results, we begin by defining a K–dimensional process $\gamma^k \equiv \{\gamma_t^k, t \geq 0\}$, $1 \leq k \leq K$, by

$$\gamma_t^k = \inf\nolimits_{0\leq s\leq t}\zeta_t^k, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{4.1}$$

The K–dimensional reflected diffusion process $(\eta^1, \ldots, \eta^K)$ is then given by

$$\eta_t^k = \zeta_t^k - \gamma_t^k, \quad 1 \leq k \leq K, \quad t \geq 0. \tag{4.2}$$

Finally, let $\{\mathcal{F}_t,\ t \geq 0\}$ be the filtration associated with the $(K+1)$–dimensional Brownian motion $(\xi^0, \ldots, \xi^K)$, i.e., $\mathcal{F}_t := \sigma - \{\xi_s^k,\ 0 \leq s \leq t,\ 0 \leq k \leq K\}$ for each $t \geq 0$.

Theorem 4.1 *We have*

(i) : The process $\{(\zeta_t, \gamma_t),\ t \geq 0\}$ is a stationary $(P, \mathcal{F}_t)$–Markov process;

(ii) : The process $\{\eta_t,\ t \geq 0\}$ is a $(P, \mathcal{F}_t)$–Markov process;

Proof. Fix $t \geq 0$ and $h > 0$. For each $1 \leq k \leq K$, we notice that

$$\begin{aligned}
\gamma_{t+h}^k - \gamma_t^k &= \inf\{\gamma_t^k, \inf\nolimits_{t<s\leq t+h}\zeta_t^k\} - \gamma_t^k \\
&= \inf\{0, \inf\nolimits_{t<s\leq t+h}\zeta_t^k - \gamma_t^k\} \\
&= \inf\{0, \zeta_t^k - \zeta_t^k + \inf\nolimits_{t<s\leq t+h}\zeta_t^k - \gamma_t^k\}
\end{aligned}$$

With this in mind, we set

ζ_t^k

$$\Phi_h(y, z; x) := E[\exp i \sum_{k=1}^{K} [y^k(\zeta_{t+h}^k - \zeta_t^k) + z^k \inf\{0, x^k + \inf\nolimits_{t<s\leq t+h}\zeta_t^k - \cancel{\gamma_t^k}\}]] \tag{4.3}$$

We now recall thatγ_t and ζ_t are $\mathcal{F}_t$–measurable. Furthermore, from basic properties of Brownian motion, we conclude that the increment process $\{\zeta_s - \zeta_t,\ t < s\}$ is independent of the σ–field $\mathcal{F}_t$ and statistically indistinguishable from ζ. from these remaarks, we now conclude that

$$E[\exp i\,[y'\zeta_{t+h} + z'\gamma_{t+h}]|\mathcal{F}_t] = \exp i\,[y'\zeta_t + z'\gamma_t] \cdot \Phi_h(y, z; \zeta_t - \gamma_t) \tag{4.4}$$

and the proof of (i) is now complete. To obtain (ii), write (4.4) with $y = -z$ so that

$$E[\exp iz'\eta_{t+h}|\mathcal{F}_t] = \exp iz'\eta_t \cdot \Phi_h(z, -z; \eta_t)$$

upon making use of (4.2). ∎

4.2. The stationary distribution

We now provide a necassary and sufficient condition for the diffusion $(\eta^1, \ldots, \eta^K)$ to have a stationary disyribution.

Proposition 4.1 *The condition $c_k > 0, 1 \le k \le K$, is necessary and sufficient to ensure that the K– dimensional process $(\eta^1, \ldots, \eta^K)$ converges in distribution to a proper vector $(\eta^1_\infty, \ldots, \eta^K_\infty)$ as $t \uparrow \infty$.*

Proof. For each $r = 1, 2 \ldots$, set

$$\tilde{W}^k_n(r) := \max_{0 \le i \le n} S^k_i(r), \quad 1 \le k \le K \tag{4.5}$$

and observe that

$$(W^1_n(r), \ldots, W^K_n(r)) =_{st} (\tilde{W}_n(r), \ldots, \tilde{W}^K_n(r)). \qquad n = 0, 1 \ldots (4.6)$$

For $r = 1, 2 \ldots$, we now define the stochastic processes $\tilde{\mu}^k(r) \equiv \{\tilde{\mu}^k_t(r), t \ge 0\}, 1 \le k \le K$, by

$$\begin{aligned} \tilde{\mu}^k_t(r) &= \frac{\tilde{W}^k_{[rt]}(r)}{\sqrt{r}} \\ &= \max_{0 \le s \le t} \frac{S^k_{[rs]}(r)}{\sqrt{r}} \\ &= \max_{0 \le s \le t} \zeta^k_s(r), \quad 1 \le k \le K, \quad t \ge 0. \end{aligned} \tag{4.7}$$

Also define the stochastic processes $\tilde{\mu}^k \equiv \{\tilde{\mu}_t^k, t \geq 0\}, 1 \leq k \leq K$, by

$$\tilde{\mu}_t^k := \sup_{0 \leq s \leq t} \zeta_s^k, \quad 1 \leq k \leq K, \quad t \geq 0.$$

Using Lemma 3.2 we conclude that as $r \uparrow \infty$,

$$(\tilde{\mu}^1(r), \ldots, \tilde{\mu}^K(r)) \Rightarrow (\tilde{\mu}^1, \ldots, \tilde{\mu}^K) \tag{4.8}$$

in $D[0,\infty)^K$, and in particular

$$(\tilde{\mu}_t^1(r), \ldots, \tilde{\mu}_t^K(r)) \xrightarrow{\mathbb{D}} (\tilde{\mu}_t^1, \ldots, \tilde{\mu}_t^K), \quad t \geq 0. \tag{4.9}$$

However, (4.6) is equivalent to

$$(\tilde{\mu}_t^1(r), \ldots, \tilde{\mu}_t^K(r)) =_{st} (\mu_t^1(r), \ldots, \mu_t^K(r)), \quad r = 1, 2 \ldots, \quad t \geq 0 \tag{4.10}$$

and (4.9) thus implies that as $r \uparrow \infty$,

$$(\mu_t^1(r), \ldots, \mu_t^K(r)) \xrightarrow{\mathbb{D}} (\tilde{\mu}_t^1, \ldots, \tilde{\mu}_t^K), \quad t \geq 0. \tag{4.11}$$

Consequently,

$$(\eta_t^1, \ldots, \eta_t^K) =_{st} (\tilde{\mu}_t^1, \ldots, \tilde{\mu}_t^K), \quad t \geq 0 \tag{4.12}$$

upon invoking Lemma 3.3.

The monotonocity of the sample paths $t \to \tilde{\mu}_t^k, 1 \leq k \leq K$, yields the convergence

$$\tilde{\mu}_t^k \uparrow \tilde{\mu}_\infty^k := \sup_{t \geq 0} \zeta_t^k, \quad 1 \leq k \leq K \tag{4.13}$$

as $t \uparrow \infty$, hence by (4.12)

$$(\eta_t^1, \ldots, \eta_t^K) \xrightarrow{\mathbb{D}} (\tilde{\mu}_\infty^1, \ldots, \tilde{\mu}_\infty^K) \tag{4.14}$$

as $t \uparrow \infty$.

It is well known [13] that $\tilde{\mu}_\infty^k < \infty$ a.s. iff $c_k > 0, 1 \leq k \leq K$, so that $(\eta^1, \ldots, \eta^K)$ has a stationary distribution iff $c_k > 0, 1 \leq k \leq K$. ∎

In order to obtain heavy traffic approximations for the fork–join queue, we have to solve for the stationary distribution of the limiting diffusion κ. This in general is a difficult problem due to the fact that κ is the maximum of K diffusions η^k, that are correlated with one another. For all $t \geq 0$ and $1 \leq k \leq K$, it is well known [13] that the marginal distribution of each RV $\eta_t^k = g(\zeta^k)_t$ is given by

$$I\!P(\eta_t^k \leq x) = \Phi\left(\frac{x + c_k t}{\sqrt{\sigma_k^2 + \sigma_0^2 t}}\right) - e^{-\frac{2c_k x}{\sigma_k^2+\sigma_0^2}} \Phi\left(\frac{-x + c_k t}{\sqrt{\sigma_k^2 + \sigma_0^2 t}}\right), \quad x \geq 0.$$

However we do not know the joint distribution of the vector $(\eta_t^1, \ldots, \eta_t^K)$ due to the correlation that exists between the different components. Hence, since the distribution of κ_t depends upon this joint distribution, we are unable to evaluate it directly.

The traditional method of overcoming this difficulty is by deriving a partial differential equation (with appropriate boundary conditions) that the joint distribution satisfies. We explore this option in the next section. In the Section 4 we obtain diffusions that bound the limiting diffusion for the end–to–end delay from above and from below in the sense of stochastic ordering. The significant fact is that the stationary distributions for the bounding diffusions can be easily obtained and they serve to bound the stationary distribution of the original diffusion.

5. A PDE for the stationary distribution

It was shown in the last section that $(\eta^1, \ldots, \eta^K)$ forms a K dimensional Markov process. The next step is to obtain the Fokker–Plank equations that are satisfied by the stationary distribution of this process. The process of deriving this PDE is exactly the same as given by Harrison and Reiman [11] in the context of two single server queues with symmetric routing. Here we just give the final PDE and the reader may consult [11] for further details.

Define the following matrix.

$$\Pi = \begin{pmatrix} \sigma_1^2 + \sigma_0^2 & 2\sigma_0^2 & \cdots & 2\sigma_0^2 \\ 2\sigma_0^2 & \sigma_2^2 + \sigma_0^2 & \cdots & 2\sigma_0^2 \\ \vdots & \vdots & \ddots & \vdots \\ 2\sigma_0^2 & 2\sigma_0^2 & \cdots & \sigma_K^2 + \sigma_0^2 \end{pmatrix}. \tag{5.1}$$

Let $\pi(z_1, \ldots, z_K)$ be the stationary density of the process $(\eta^1, \ldots, \eta^K)$. Then π satisfies the following PDE.

$$\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} R_{ij}\frac{\partial^2 \pi}{\partial z^i \partial z^j} + \sum_{j=1}^{K} c_j \frac{\partial \pi}{\partial z^j} = 0, \tag{5.2a}$$

$$\sum_{j=1}^{K} \Pi_{ij}\frac{\partial \pi}{\partial z^j} - 2c_i\pi = 0 \text{ if } z_i = 0, \tag{5.2b}$$

$$\int_{\mathbb{R}_+^K} \pi(z)dz = 1. \tag{5.2c}$$

Hence in order to obtain the stationary distribution π for the limiting diffusion, we have to solve a second order elliptic PDE in K independent variables (5.2a) and oblique boundary conditions (5.2b). We obtain the solution for a special case when $K = 2$, in Section 7.

6. Bounds for the stationary distribution

In the last section we exhibited a PDE for the stationary distribution π of the limiting diffusion. In general this PDE is difficult to solve and in this section our objective is to obtain some additional useful information about the stationary distribution by bounding the limiting diffusion from above and from below by two other diffusions whose stationary distribution is easier to characterize. The lower bound is obtained by using by using ideas from stochastic ordering theory, while the upper bound is obtained by using the concept of associated RVs. These results constitute an extension to the continuous time case of results that originally appeared in [2]. Appendix A contains a collection of the basic definitions and results regarding stochastic ordering theory that we shall use.

6.1 A lower bound

It is a well–known fact that for certain queueing systems operating in their stable regime, determinism in either the arrival or the service processes minimizes queueing delays. Our results in this section imply that this property continues to hold for the limiting diffusion of the end–to–end delay of the Fork–Join queue in heavy traffic. We prove this result by working directly with the limiting diffusion.

Recall that for each $1 \leq k \leq K$, $t \geq 0$, we have

$$\zeta_t^k = \sigma_k \xi_t^k - \sigma_0 \xi_t^0 - c_k t, \quad \eta_t^k = g(\zeta^k)_t \tag{6.1a}$$

and

$$\kappa_t = \max_{1 \leq k \leq K} \eta_t^k. \tag{6.1b}$$

We now construct a new limiting diffusion for the Fork–Join system which is the same as the original one, except for the absence of the Wiener process ξ^0~~, which no longer appears in the equations~~. The intuitive reason for this may be understood as follows: The stochastic process $\xi^0(r)$, obtained after appropriately scaling a deterministic ~~input~~ arrival sequence, converges to 0 as $r \uparrow \infty$, instead of to a Wiener process as was formerly the case. We shall use the same notation to denote ~~quantities~~ the processes in the new system except that we shall underline them.
For $1 \leq k \leq K$, $t \geq 0$, we define

The corresponding quantities in the modified system ~~are~~,

$$\underline{\zeta}_t^k = \sigma_k \xi_t^k - c_k t, \quad \underline{\eta}_t^k = g(\underline{\zeta}^k)_t \tag{6.2a}$$

and

$$\underline{\kappa}_t = \max_{1 \leq k \leq K} \underline{\eta}_t^k. \tag{6.2b}$$

We now present our first result.

Combine with Theorem 6.1

Lemma 6.1 *Let $\underline{I\!\!F}$ be the $\sigma-$ field of events generated on the sample space Ω by the stochastic process $(\xi^1, \ldots \xi^K)$. The inequalities*

$$\underline{\eta}_t^k \leq I\!\!E[\eta_t^k \mid \underline{I\!\!F}], \quad 1 \leq k \leq K, \quad t \geq 0 \tag{6.3}$$

hold, whence

$$\underline{\kappa}_t \leq I\!\!E[\kappa_t \mid \underline{I\!\!F}], \quad t \geq 0 \tag{6.4}$$

Proof. For each $1 \leq k \leq K$ and $t \geq 0$, we have

$$\begin{aligned} \eta_t^k &= g(\zeta^k)_t \\ &= \sup_{0 \leq s \leq t} (\zeta_t^k - \zeta_s^k). \end{aligned}$$

Since

$$\eta_t^k = \sup_{0 \le s \le t} (\zeta_t^k - \zeta_s^k) \ge \zeta_t^k - \zeta_s^k, \quad 1 \le k \le K, \quad 0 \le s \le t,$$

we readily conclude that

$$\mathbb{E}(\eta_t^k \mid \underline{S}) \ge \mathbb{E}(\zeta_t^k \mid \underline{S}) - \mathbb{E}(\zeta_s^k \mid \underline{S}), \quad 1 \le k \le K, \quad 0 \le s \le t$$

so that

$$\begin{aligned} \mathbb{E}(\eta_t^k \mid \underline{S}) &\ge \sup_{0 \le s \le t} \left[\mathbb{E}(\zeta_t^k \mid \underline{S}) - \mathbb{E}(\zeta_s^k \mid \underline{S}) \right] \\ &= \sup_{0 \le s \le t} (\sigma_k \xi_t^k - \sigma_0 \mathbb{E}\xi_t^0 - c_k t - \sigma_k \xi_s^k + \sigma_0 \mathbb{E}\xi_s^0 + c_k s). \end{aligned}$$

Since $\mathbb{E}\xi_t^0 = 0$ for all $t \ge 0$, we get

$$\begin{aligned} \mathbb{E}(\eta_t^k \mid \underline{S}) &\ge \sup_{0 \le s \le t} (\sigma_k \xi_t^k - c_k t - \sigma_k \xi_s^k + c_k s) \\ &= \sup_{0 \le s \le t} (\underline{\zeta}_t^k - \underline{\zeta}_s^k) \\ &= \underline{\eta}_t^k, \quad 1 \le k \le K, \quad t \ge 0 \end{aligned}$$

and this proves (4.3). In order to prove (4.4) we note that

$$\mathbb{E}(\kappa_t \mid \underline{S}) \ge \max_{1 \le k \le K} \mathbb{E}(\eta_t^k \mid \underline{S}), \quad t \ge 0$$

and (4.3) now implies

$$\mathbb{E}(\kappa_t \mid \underline{S}) \ge \max_{1 \le k \le K} \underline{\eta}_t^k = \underline{\kappa}_t, \quad t \ge 0.$$

∎

Theorem 6.1 *The following inequality holds*

$$\underline{\kappa}_t \le_{icx} \kappa_t, \quad t \ge 0. \tag{6.5}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex non-decreasing function. By Lemma 4.1, we have

$$f(\mathbb{E}(\kappa_t \mid \underline{\mathbb{S}})) \geq f(\underline{\kappa}_t), \quad t \geq 0.$$

Therefore, upon applying Jensen's inequality, we see that

$$\mathbb{E}f(\kappa_t) \geq \mathbb{E}f(\underline{\kappa}_t), \quad t \geq 0$$

and (6.5) now follows. ∎

According to Proposition 4.1, the condition $c_k > 0, 1 \leq k \leq K$, is sufficient to ensure that the RVs $\underline{\kappa}_t$, and κ_t converge as $t \uparrow \infty$ to proper RVs $\underline{\kappa}_\infty$, and κ_∞ respectively. Our next result shows that the the RVs $\underline{\kappa}_\infty$ and κ_∞ continue to satisfy (6.5).

Theorem 6.2 *Under the condition $c_k > 0, 1 \leq k \leq K$, the following inequality holds*

$$\underline{\kappa}_\infty \leq_{icx} \kappa_\infty. \tag{6.6}$$

Proof. Recall from the discussion in Section 4.2 that

$$\kappa_t =_{st} \tilde{\kappa}_t \ \text{ and } \ \underline{\kappa}_t =_{st} \tilde{\underline{\kappa}}_t, \quad t \geq 0 \tag{6.7}$$

where

$$\tilde{\kappa}_t = \max_{1 \leq k \leq K} \tilde{\eta}_t^k \ \text{ with } \ \tilde{\eta}_t^k = \sup_{0 \leq s \leq t} \zeta_s^k, \quad 1 \leq k \leq K, \quad t \geq 0$$

and

$$\tilde{\underline{\kappa}}_t = \max_{1 \leq k \leq K} \tilde{\underline{\eta}}_t^k \ \text{ with } \ \tilde{\underline{\eta}}_t^k = \sup_{0 \leq s \leq t} \underline{\zeta}_s^k, \quad 1 \leq k \leq K, \quad t \geq 0.$$

From (6.5) and (6.7) we conclude that for all convex non-decreasing functions f, we have

$$\mathbb{E}f(\tilde{\underline{\kappa}}_t) \leq \mathbb{E}f(\tilde{\kappa}_t), \quad t \geq 0. \tag{6.8}$$

Since the RVs $\tilde{\eta}_t^k$ and $\tilde{\underline{\eta}}_t^k$ for $1 \leq k \leq K$, are non-decreasing with t, it follows that the RVs $\tilde{\kappa}_t$ and $\tilde{\underline{\kappa}}_t$ are also non-decreasing with t. An application of the monotone convergence theorem now ensures that

$$\mathbb{E}f(\tilde{\underline{\kappa}}_\infty) \leq \mathbb{E}f(\tilde{\kappa}_\infty), \tag{6.9}$$

from which (6.6) is now immediate. ∎

6.3 Upper bounds by association

By using the concept of associated stochastic processes, we exhibit a family of diffusions that bound the diffusion for the end–to–end delay in the sense of strong stochastic ordering.

Lemma 6.2 *For each $t \geq 0$, the RVs $\{\eta_t^k, \ldots, \eta_t^K\}$, are associated.*

Proof. In order to prove this property we use Lemma A2 from Appendix A. First note that the RVs

$$\{\xi_t^1 - \xi_s^1, \ldots, \xi_t^K - \xi_s^K, -(\xi_t^0 - \xi_s^0)\}, \quad 0 \leq s \leq t$$

are independent and hence are associated by property (i). By property (iv) the RVs

$$\{\sigma_k[\xi_t^k - \xi_s^k] - \sigma_A[\xi_t^0 - \xi_s^0] - c_k(t-s), 1 \leq k \leq K\}, \quad 0 \leq s \leq t$$

are associated, i.e., the RVs

$$\{\zeta_t^1 - \zeta_s^1, \ldots, \zeta_t^K - \zeta_s^K\}, \quad 0 \leq s \leq t \tag{6.10}$$

are associated.

Fix $t \geq 0$. Define the set $\mathbb{D}_t$ by

$$\mathbb{D}_t = \{\frac{kt}{n}, 0 \leq k \leq n, n = 1, 2 \ldots\}$$

and note that $\mathbb{D}_t$ is a countable dense subset of $[0, t]$. Since the process ζ is separable, it follows [6] that

$$\sup_{0 \leq s \leq t} (\zeta_t^k - \zeta_s^k) = \max_{s \in \mathbb{D}_t} (\zeta_t^k - \zeta_s^k), \quad 1 \leq k \leq K. \tag{6.11}$$

For each $n = 1, 2 \ldots$, define the sets of RVs $A_k^n, 1 \leq k \leq n$, by

$$A_k^n = \{\zeta_{\frac{kt}{n}}^1 - \zeta_{\frac{(k-1)t}{n}}^1, \ldots, \zeta_{\frac{kt}{n}}^K - \zeta_{\frac{(k-1)t}{n}}^K\}, \quad 1 \leq k \leq n.$$

By (6.10) it follows that the RVs in each $A_k^n, 1 \leq k \leq n$, are associated. But since the processes ζ has independent increments, it follows from property (ii) that the RVs in $A^n = A_1^n \cup \ldots \cup A_n^n$ are associated. By taking sums of RVs in A^n it follows that the RVs

$$\{\zeta_t^1, \ldots, \zeta_t^K,$$

$$\zeta_t^1 - \zeta_{\frac{t}{n}}^1, \ldots, \zeta_t^K - \zeta_{\frac{t}{n}}^K, \ldots$$

$$\ldots, \zeta_t^1 - \zeta_{\frac{n-1}{n}t}^1, \ldots, \zeta_t^K - \zeta_{\frac{n-1}{n}t}^K\}, \quad n = 1, 2 \ldots, \quad t \geq 0$$

are associated. Another application of property (iv) assures us that the RVs

$$\{\max_{0 \leq k \leq n}(\zeta_t^1 - \zeta_{\frac{kt}{n}}^1), \ldots, \max_{0 \leq k \leq n}(\zeta_t^K - \zeta_{\frac{kt}{n}}^K)\}, \quad n = 1, 2 \ldots, \quad t \geq 0$$

are associated. Letting $n \uparrow \infty$ it follow that the RVs

$$\{\max_{s \in \mathbb{D}_t}(\zeta_t^1 - \zeta_s^1), \ldots, \max_{s \in \mathbb{D}_t}(\zeta_t^K - \zeta_s^K)\}, \quad t \geq 0$$

are associated. Finally from (6.11) we conclude that the RVs

$$\{\max_{0 \leq s \leq t}(\zeta_t^1 - \zeta_s^1), \ldots, \max_{0 \leq s \leq t}(\zeta_t^K - \zeta_s^K)\}, \quad t \geq 0$$

are associated, just another way of saying that the RVs $\{\eta_t^1, \ldots, \eta_t^K\}$ are associated.

∎

We now define the stochastic processes $\overline{\eta}^k \equiv \{\overline{\eta}_t^k, t \geq 0\}, 1 \leq k \leq K$, which form independent versions of the stochastic processes $(\eta^1, \ldots \eta^K)$ in the sense of Definition A3. For this purpose define K additional independent Wiener processes $\xi^{0,1}, \ldots, \xi^{0,K}$. For $1 \leq k \leq K$, and $t \geq 0$, define

$$\overline{\zeta}_t^k = \sigma_k \xi_t^k - \sigma_0 \xi_t^{0,k} - c_k t, \tag{6.12a}$$

$$\overline{\eta}_t^k = g(\overline{\zeta}^k)_t \quad \text{and} \quad \overline{\kappa}_t = \max_{1 \leq k \leq K} \overline{\eta}_t^k. \tag{6.12b}$$

From Lemma 6.2 and Lemma A1, it directly follows that

$$\kappa_t = \max_{1 \leq k \leq K} \eta_t^k \leq_{st} \max_{1 \leq k \leq K} \overline{\eta}_t^k = \overline{\kappa}_t$$

This is stated in the next result.

Theorem 6.3 *The following relation holds true for $t \geq 0$,*

$$\kappa_t \leq_{st} \overline{\kappa}_t. \tag{6.13}$$

It was shown in the last chapter that the condition $c_k > 0, 1 \leq k \leq K$, is necessary and sufficient to ensure that the RVs κ_t and $\overline{\kappa}_t$ converge weakly as $t \uparrow \infty$ to proper RVs κ_∞ and $\overline{\kappa}_\infty$, respectively. The following result is then an immediate consequence of Theorem 6.3 and Proposition 1.2.3 in [27].

Theorem 6.4 *Under the condition $c_k > 0, 1 \leq k \leq K$, the following inequality holds*

$$\kappa_\infty \leq_{st} \overline{\kappa}_\infty. \tag{6.14}$$

6.4 Some computations

In this section we carry out explicit calculations of the distributions of the bounding diffusions in the transient as well the stationary case. We shall denote as a symmetric Fork–Join queue, the one in which all the K service times have identical probability distribution functions, so that

$$c_1 = c_2 = \ldots = c_K = c, \text{ and } \sigma_1 = \sigma_2 = \ldots = \sigma_K = \sigma$$

We also use the notation $H_K, K = 1, 2\ldots$ for the partial sums of the Harmonic series, i.e.,

$$H_K = \sum_{k=1}^{K} \frac{1}{k}. \qquad K = 1, 2\ldots$$

Lower bounds are computed in Section 6.4.1, while upper bounds are computed in Section 6.4.2.

6.4.1 Lower Bounds

The results of Theorem 6.1 and Theorem 6.2 imply that

$$\mathbb{E}\underline{\kappa}_t \leq \mathbb{E}\kappa_t, \quad t \geq 0$$

and

$$\mathbb{E}\underline{\kappa}_\infty \leq \mathbb{E}\kappa_\infty.$$

Our objective in this section is to give explicit formulae for $\mathbb{E}\underline{\kappa}_t$ and $\mathbb{E}\underline{\kappa}_\infty$.

We first proceed with the calculation of $\mathbb{E}\underline{\kappa}_t$. Recall that

$$\underline{\kappa}_t = \max_{1 \leq k \leq K} \underline{\eta}_t^k, \quad t \geq 0$$

where $\underline{\eta}_t^k = g(\sigma_k \xi^k - c_k)_t, t \geq 0, 1 \leq k \leq K$. Note that since the stochastic processes $\xi^k, 1 \leq k \leq K$, are independent, it follows that the stochastic processes $\underline{\eta}^k, 1 \leq k \leq K$ are also independent, and therefore

$$\mathbb{P}(\underline{\kappa}_t \leq z) = \prod_{k=1}^{K} \mathbb{P}(\underline{\eta}_t^k \leq z), \quad t \geq 0 \tag{6.15}$$

for all $z \geq 0$. Note that $\underline{\eta}^k, 1 \leq k \leq K$, are diffusion processes with drift $-c_k$ and variance σ_k, which are reflected from the origin. The transient distribution for these processes is well known [13]. For all $z \geq 0$,

$$\mathbb{P}(\underline{\eta}_t^k \leq z) = \Phi\left(\frac{z + c_k t}{\sigma_k \sqrt{t}}\right) - e^{-\frac{2c_k z}{\sigma_k^2}} \Phi\left(\frac{-z + c_k t}{\sigma_k \sqrt{t}}\right), \quad t \geq 0. \tag{6.16}$$

From (6.15) and (6.16) it follows that for all $z \geq 0$,

$$\mathbb{P}(\underline{\kappa}_t \leq z) = \prod_{k=1}^{K} \left[\Phi\left(\frac{z + c_k t}{\sigma_k \sqrt{t}}\right) - e^{-\frac{2c_k z}{\sigma_k^2}} \Phi\left(\frac{-z + c_k t}{\sigma_k \sqrt{t}}\right)\right], \quad t \geq 0. \tag{6.17}$$

Since

$$\mathbb{E}\underline{\kappa}_t = \int_0^\infty [1 - \prod_{k=1}^{K} \mathbb{P}(g(\underline{\zeta}^k)_t \leq z)] dz, \quad t \geq 0 \tag{6.18}$$

it follows that

$$\mathbb{E}\underline{\kappa}_t = \int_0^\infty \left[1 - \prod_{k=1}^K \left(\Phi\left(\frac{z + c_k t}{\sigma_k\sqrt{t}}\right) - e^{-\frac{2c_k z}{\sigma_k^2}}\Phi\left(\frac{-z + c_k t}{\sigma_k\sqrt{t}}\right)\right)\right] dz, \quad t \geq 0. \tag{6.19}$$

In the symmetric case, equations (6.17) and (6.19) reduce for all $z \geq 0$ to

$$\mathbb{P}(\underline{\kappa}_t \leq z) = \left[\Phi\left(\frac{z + ct}{\sigma\sqrt{t}}\right) - e^{-\frac{2cz}{\sigma^2}}\Phi\left(\frac{-z + ct}{\sigma\sqrt{t}}\right)\right]^K, \quad t \geq 0 \tag{6.20}$$

and

$$\mathbb{E}\underline{\kappa}_t = \int_0^\infty \left[1 - \left(\Phi\left(\frac{z + ct}{\sigma\sqrt{t}}\right) - e^{-\frac{2cz}{\sigma^2}}\Phi\left(\frac{-z + ct}{\sigma\sqrt{t}}\right)\right)^K\right] dz, \quad t \geq 0. \tag{6.21}$$

We now proceed with the calculation of $\mathbb{E}\underline{\kappa}_\infty$ under the condition $c_k > 0, 1 \leq k \leq K$. Recalling that

$$\underline{\kappa}_\infty = \max_{1 \leq k \leq K} \underline{\eta}_\infty^k,$$

we see from (6.16) that

$$\mathbb{P}(\underline{\eta}_\infty^k \leq z) = 1 - e^{-\frac{2c_k z}{\sigma_k^2}}, \quad z \geq 0 \tag{6.22}$$

so that

$$\mathbb{P}(\underline{\kappa}_\infty \leq z) = \prod_{k=1}^K [1 - e^{-\frac{2c_k z}{\sigma_k^2}}], \quad z \geq 0 \tag{6.23}$$

Hence $\mathbb{E}\underline{\kappa}_\infty$ is given by

$$\mathbb{E}\underline{\kappa}_\infty = \int_0^\infty [1 - \prod_{k=1}^K (1 - e^{-\frac{2c_k z}{\sigma_k^2}})] dz \tag{6.24}$$

It is clear that

$$1 - \prod_{k=1}^K (1 - e^{-\frac{2c_k z}{\sigma_k^2}}) = \sum_{k=1}^K (-1)^{k+1} \sum_{I \in \mathbb{I}_k} e^{-\sum_{k \in I} \frac{2c_k z}{\sigma_k^2}} \tag{6.25}$$

where

$$\mathbb{I}_k = \{I \subseteq \{1, \ldots, K\} :| I |= k\}, \quad 1 \leq k \leq K.$$

For any non-empty subset I of $\{1, \ldots, K\}$, we see that

$$\int_0^\infty e^{-\sum_{k\in I} \frac{2c_k z}{\sigma_k^2}} dz = (\sum_{k\in I} \frac{2c_k}{\sigma_k^2})^{-1} \tag{6.26}$$

so that

$$\mathbb{E}\underline{\kappa}_\infty = \sum_{k=1}^{K} (-1)^{k+1} \sum_{I\in \mathbb{I}_k} (\sum_{k\in I} \frac{2c_k}{\sigma_k^2})^{-1}. \tag{6.27}$$

In the symmetric case, equations (6.23) and (6.24) reduce to

$$\mathbb{P}(\underline{\kappa}_\infty \leq z) = [1 - e^{-\frac{2cz}{\sigma^2}}]^K, \quad z \geq 0 \tag{6.28}$$

and

$$\mathbb{E}\underline{\kappa}_\infty = \int_0^\infty [1 - e^{-\frac{2cz}{\sigma^2}}]^K dz. \tag{6.29}$$

Taking note of the fact that $| \mathbb{I} |= \binom{K}{k}$ and of the identity

$$\sum_{k=1}^{K} \frac{(-1)^{k+1}}{k} \binom{K}{k} = \sum_{k=1}^{K} \frac{1}{k} = H_K, \qquad K = 1, 2 \ldots$$

equation (6.29) reduces to

$$\mathbb{E}\underline{\kappa}_\infty = \frac{\sigma^2}{2c} H_K. \tag{6.30}$$

For the homogeneous case it is also possible to obtain explicit formulae for higher moments of $\underline{\kappa}_\infty$ and we proceed to do so next. Note that the density $f(z), z \geq 0$ of $\underline{\kappa}_\infty$ is obtained by differentiating (6.28), so that

$$f(z) = \frac{2cK}{\sigma^2} e^{-\frac{2cz}{\sigma^2}} [1 - e^{-\frac{2cz}{\sigma^2}}]^{K-1}$$

Hence

$$\mathbb{E}\underline{\kappa}_\infty^n = \frac{2cK}{\sigma^2} \int_0^\infty z^n e^{-\frac{2cz}{\sigma^2}} [1 - e^{-\frac{2cz}{\sigma^2}}]^{K-1} dz$$

some.

After making ~~a~~ simplifications and integrating, we obtain the following formula for the n^{th} moment, of $\underline{K}_\infty$.

$$\mathbb{E}\underline{\kappa}_\infty^n = n!(\frac{\sigma^2}{2c})^n \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k+1}}{k^n} \qquad n = 1, 2 \ldots (6.31)$$

6.4.2 Upper Bounds

The results of Theorem 6.3 and Theorem 6.4 imply that

$$\mathbb{E}\overline{\kappa}_t \geq \mathbb{E}\kappa_t, \quad t \geq 0$$

and

$$\mathbb{E}\overline{\kappa}_\infty \geq \mathbb{E}\kappa_\infty.$$

Our objective in this section is to give explicit formulae for $\mathbb{E}\overline{\kappa}_t$ and $\mathbb{E}\overline{\kappa}_\infty$. Since all the calculations involved are exactly the same as in the last section, we only give the final formulae in each case.

Proceeding exactly as in the last section, it is possible to show that

$$\mathbb{E}\overline{\kappa}_t = \int_0^\infty \left[1 - \prod_{k=1}^{K} \left(\Phi\left(\frac{z + c_k t}{\sqrt{(\sigma_k^2 + \sigma_0^2)t}} \right) - e^{-\frac{2c_k z}{\sigma_k^2 + \sigma_0^2}} \Phi\left(\frac{-z + c_k t}{\sqrt{(\sigma_k^2 + \sigma_0^2)t}} \right) \right) \right] dz,$$

$$t \geq 0 \quad (6.32)$$

and in the symmetric case,

$$\mathbb{E}\overline{\kappa}_t = \int_0^\infty \left[1 - \left(\Phi\left(\frac{z + ct}{\sqrt{(\sigma^2 + \sigma_0^2)t}} \right) - e^{-\frac{2cz}{\sigma^2 + \sigma_0^2}} \Phi\left(\frac{-z + ct}{\sqrt{(\sigma^2 + \sigma_0^2)t}} \right) \right)^K \right] dz,$$

$$t \geq 0. \quad (6.33)$$

Under the condition $c_k > 0, 1 \leq k \leq K$, we further have that

$$\mathbb{E}\overline{\kappa}_\infty = \sum_{k=1}^{K} (-1)^{k+1} \sum_{I \in \mathbb{I}_k} (\sum_{k \in I} \frac{2c_k}{\sigma_k^2 + \sigma_0^2})^{-1} \qquad (6.34)$$

and in the symmetric case

$$\mathbb{E}\overline{\kappa}_\infty = \frac{\sigma^2 + \sigma_0^2}{2c} H_K. \tag{6.35}$$

moment

As in the last section, we can obtain the following formula for the n^{th} of $\overline{\kappa}_\infty$,

$$\mathbb{E}\overline{\kappa}_\infty^n = n!(\frac{\sigma^2 + \sigma_0^2}{2c})^n \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k+1}}{k^n} \tag{6.36}$$

Equations (6.30) and (6.35) imply that in the symmetric case

$$\frac{\sigma^2}{2c} H_K \leq \mathbb{E}\kappa_\infty \leq \frac{\sigma^2 + \sigma_0^2}{2c} H_K, \tag{6.37}$$

and (6.31) and (6.36) imply that

$$n!(\frac{\sigma^2}{2c})^n \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k+1}}{k^n} \leq \mathbb{E}\kappa_\infty^n \leq n!(\frac{\sigma^2 + \sigma_0^2}{2c})^n \sum_{k=1}^{K} \binom{K}{k} \frac{(-1)^{k+1}}{k^n} \quad n = 2, 3 \ldots \tag{6.38}$$

Note that since

$$\log(K+1) \leq H_K \leq \log K$$

it follows from (6.37) that the expectation of the normalized end–to–end delay of a symmetric Fork–Join queue in heavy traffic, increases logarithmically with K.

Equation (6.37) reveals an interesting difference between the asymptotic behavior in K, for Fork–Join queues operating in heavy traffic with those operating in their stable regime. It was shown in [2] that moments of the end–to–end delay of a stable Fork–Join queue increase logarithmically in K provided the following condition is satisfied;

Let $A^*(s)$ and $B^*(s)$ denote the Laplace-Stieltjes transform of the interarrival and service times. The transform $B^*(s)$ is assumed to be rational so that the function $s \to f(s)$ which is initially defined for $\mathbf{R}e(s) = 0$ by

$$f(s) = A^*(s)B^*(-s)$$

is continuable in the region $\mathbf{R}e(s) \geq 0$.

Under this assumption it is shown in [2] that the response time of each queue has an exponential tail, which leads to the logarithmic behavior. However in heavy traffic the response times *always* have an exponential tail provided they satisfy assumptions **(Ia)–(Ic)** from Section 3. Hence even those Fork–Join queues whose end–to–end delay does not grow logarithmically with K when they are in their stable regime, (since they do not satisfy the above assumption), exhibit logarithmic growth of their end–to–end delay with K, once they are in heavy traffic. The inequality (6.37) has also proven to be useful in the comparison of the Fork–Join queue with alternative queues with different scheduling strategies for the tasks [31].

7.1 Solution to the PDE

In this section our objective is to obtain a solution to the PDE for the stationary density of the diffusion process for the queue delays in a Fork–Join queue~~, which was derived in Section 3.~~ From this stationary density we can then recover some heavy traffic information about the queue. We only consider the solution of the PDE for the case of two independent variables, so that the results of this chapter are applicable to two dimensional Fork–Join systems. The technique that we shall use for solving the PDE is similar to the one used by Harrison [10] and Foschini [8] in the context of a system of single server queues in tandem. For this technique to be applicable, it is necessary to assume that $\sigma_0 = \sigma_1 = \sigma_2$. In Section 7.3 we obtain formulae for all the moments of the diffusion for the end–to–end delay for this case. These moments are combined with light traffic results [29] in order to obtain interpolation approximations.

We use the following notation. As in the last chapter, the non–negative quadrant in the (x, y)–plane will be referred to as $\mathbb{R}^2_+$. For each $0 \leq \beta \leq 1$, the region in the first and fourth quadrants that is bounded by the lines $y = \sqrt{\frac{1+\beta}{1-\beta}}x$ and $y = -\sqrt{\frac{1+\beta}{1-\beta}}x$ will be referred to as $\mathbb{R}^2_\beta$, i.e.,

$$\mathbb{R}^2_\beta = \{(x, y) \in \mathbb{R}^2_+ : -\sqrt{\frac{1+\beta}{1-\beta}}x \leq y \leq \sqrt{\frac{1+\beta}{1-\beta}}x\}$$

7.2 The queue delay processes: Symmetrical case

We consider the problem of determining the stationary density for the waiting time processes of the Fork–Join queue in the case when $K = 2$. We start by writing down the PDE obtained in Section 5. which the stationary density satisfies. The start down [illegible] the following PDF

$$\frac{1}{2}(\sigma_1^2 + \sigma_0^2)\frac{\partial^2 \pi(x,y)}{\partial x^2} + \sigma_0^2 \frac{\partial^2 \pi(x,y)}{\partial x \partial y} + \frac{1}{2}(\sigma_2^2 + \sigma_0^2)\frac{\partial^2 \pi(x,y)}{\partial y^2}$$

$$+c_1 \frac{\partial \pi(x,y)}{\partial x} + c_2 \frac{\partial \pi(x,y)}{\partial y} = 0, \quad (x,y) \in I\!R_+^2 \tag{7.1a}$$

$$BC(x=0) : \frac{1}{2}(\sigma_1^2 + \sigma_0^2)\frac{\partial \pi(0,y)}{\partial x} + \sigma_0^2 \frac{\partial \pi(0,y)}{\partial y} + c_1 \pi(0,y) = 0 \tag{7.1b}$$

$$BC(y=0) : \sigma_0^2 \frac{\partial \pi(x,0)}{\partial x} + \frac{1}{2}(\sigma_2^2 + \sigma_0^2)\frac{\partial \pi(x,0)}{\partial y} + c_2 \pi(x,0) = 0. \tag{7.1c}$$

We further make the assumption that the two queues are identical with $\sigma_1 = \sigma_2 = \sigma$ and $c_1 = c_2 = c$, and we set $\alpha^2 = \sigma_0^2 + \sigma^2$ in what follows. The equilibrium equations then simplify to the following.

$$\frac{1}{2}\alpha^2 \frac{\partial^2 \pi(x,y)}{\partial x^2} + \sigma_0^2 \frac{\partial^2 \pi(x,y)}{\partial x \partial y} + \frac{1}{2}\alpha^2 \frac{\partial^2 \pi(x,y)}{\partial y^2}$$

$$+c \frac{\partial \pi(x,y)}{\partial x} + c \frac{\partial \pi(x,y)}{\partial y} = 0, \quad (x,y) \in I\!R_+^2 \tag{7.2a}$$

$$BC(x=0) : \frac{1}{2}\alpha^2 \frac{\partial \pi(0,y)}{\partial x} + \sigma_0^2 \frac{\partial \pi(0,y)}{\partial y} + c\pi(0,y) = 0 \tag{7.2b}$$

$$BC(y=0) : \sigma_0^2 \frac{\partial \pi(x,0)}{\partial x} + \frac{1}{2}\alpha^2 \frac{\partial \pi(x,0)}{\partial y} + c\pi(x,0) = 0. \tag{7.2c}$$

We now scale the co-ordinates according to the transformation $T_0 : (x,y) \to (x_1, y_1)$, so that so that $(x_1, y_1) = (ax, ay)$ where $a = \frac{2c}{\alpha^2}$, and we set $\beta = \frac{\sigma_0^2}{\alpha^2}$ in what follows. Denoting $\pi(\frac{x_1}{a}, \frac{y_1}{a})$ by $\pi_a(x_1, y_1)$, (7.2a)–(7.2c) can then be written as

$$\frac{\partial^2 \pi_a(x_1,y_1)}{\partial x_1^2} + 2\beta \frac{\partial^2 \pi_a(x_1,y_1)}{\partial x_1 \partial y_1} + \frac{\partial^2 \pi_a(x_1,y_1)}{\partial y_1^2}$$

$$+\frac{\partial \pi_a(x_1,y_1)}{\partial x_1}+\frac{\partial \pi_a(x_1,y_1)}{\partial y_1}=0, \quad (x_1,y_1)\in I\!R^2_+ \tag{7.3a}$$

$$BC(x_1=0): \frac{\partial \pi_a(0,y_1)}{\partial x_1}+2\beta\frac{\partial \pi_a(0,y_1)}{\partial y_1}+\pi_a(0,y_1)=0 \tag{7.3b}$$

$$BC(y_1=0): 2\beta\frac{\partial \pi_a(x_1,0)}{\partial x_1}+\frac{\partial \pi_a(x_1,0)}{\partial y_1}+\pi_a(x_1,0)=0. \tag{7.3c}$$

Since

$$\beta=\frac{\sigma_0^2}{\alpha^2}=\frac{\sigma_0^2}{\sigma_0^2+\sigma^2}, \tag{7.4}$$

the parameter β is constrained to lie in the set $[0,1]$, and we shall therefore seek solutions to (7.3a)–(7.3c) with β constrained to lie in this set.

7.3 The solution in polar co–ordinates

In this section we recast our basic equation (7.3a)–(7.3c) into the form $\nabla^2\phi=\phi$, where ∇^2 is the two–dimensional Laplacian in polar form. This is accomplished by several transformations as shown below. The development of this section is inspired by Foschini [8] and Harrison [10].

The transformation is achieved in the following five steps:

(1): We start with an exponential substitution to eliminate the drift terms. Let us introduce a new function π_1 defined by

$$\pi_1(x_1,y_1)=\pi_a(x_1,y_1)e^{-b(x_1+y_1)}, \quad (x_1,y_1)\in I\!R^2_+ \tag{7.5}$$

where $b=-\frac{1}{2(1+\beta)}$. The PDE can then be re–written as

$$\frac{\partial^2\pi_1}{\partial x_1^2}+2\beta\frac{\partial^2\pi_1}{\partial x_1\partial y_1}+\frac{\partial^2\pi_1}{\partial y_1^2}=\frac{\pi_1}{2(1+\beta)}, \quad (x_1,y_1)\in I\!R^2_+ \tag{7.6a}$$

$$BC(x_1=0): \frac{\partial\pi_1}{\partial x_1}+2\beta\frac{\partial\pi_1}{\partial y_1}+\frac{\pi_1}{2(1+\beta)}=0 \tag{7.6b}$$

$$BC(y_1=0): 2\beta\frac{\partial\pi_1}{\partial x_1}+\frac{\partial\pi_1}{\partial y_1}+\frac{\pi_1}{2(1+\beta)}=0. \tag{7.6c}$$

(2): The term with the mixed derivatives can be removed by the orthogonal transformation $T_1 : (x_1, y_1) \to (x_2, y_2)$, defined by

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad (x_1, y_1) \in I\!R^2_+. \tag{7.7}$$

This transformation maps the quadrant $I\!R^2_+$ in the (x_1, y_1)–plane into a region $I\!R^2_0$ in the first and fourth quadrants of the (x_2, y_2)–plane that is bounded by the lines $x_2 = y_2$ and $x_2 = -y_2$.

Denoting $\pi_1(T_1(x_1, y_1))$ by $\pi_2(x_2, y_2)$, we obtain the following PDE.

$$(1+\beta)\frac{\partial^2 \pi_2}{\partial x_2^2} + (1-\beta)\frac{\partial^2 \pi_2}{\partial y_2^2} = \frac{\pi_2}{2(1+\beta)}, \quad (x_2, y_2) \in I\!R^2_0 \tag{7.8a}$$

$$BC(x_2 = y_2) : (2\beta+1)\frac{\partial \pi_2}{\partial x_2} + (2\beta-1)\frac{\partial \pi_2}{\partial y_2} + \frac{\pi_2}{\sqrt{2}(1+\beta)} = 0 \tag{7.8b}$$

$$BC(x_2 = -y_2) : (2\beta+1)\frac{\partial \pi_2}{\partial x_2} - (2\beta-1)\frac{\partial \pi_2}{\partial y_2} + \frac{\pi_2}{\sqrt{2}(1+\beta)} = 0. \tag{7.8c}$$

(3): The next transformation $T_2 : (x_2, y_2) \to (x_3, y_3)$ is defined by

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+\beta}} & 0 \\ 0 & \frac{1}{\sqrt{1-\beta}} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad (x_2, y_2) \in I\!R^2_0 \tag{7.9}$$

This transformation maps the region $I\!R^2_0$ in the (x_2, y_2)–plane into a region $I\!R^2_\beta$ in the first and fourth quadrants of the (x_3, y_3)–plane that is bounded by the lines $y_3 = \sqrt{\frac{1+\beta}{1-\beta}}x_3$ and $y_3 = -\sqrt{\frac{1+\beta}{1-\beta}}x_3$. Denoting $\pi_2(T_2(x_2, y_2))$ by $\pi_3(x_3, y_3)$, we obtain the PDE

$$\nabla^2 \pi_3 = \frac{\pi_3}{2(1+\beta)}, \quad (x_3, y_3) \in I\!R^2_\beta \tag{7.10a}$$

$$BC\left(y_3 = \sqrt{\frac{1+\beta}{1-\beta}}x_3\right) : \frac{(2\beta+1)}{\sqrt{1+\beta}}\frac{\partial \pi_3}{\partial x_3} + \frac{(2\beta-1)}{\sqrt{1-\beta}}\frac{\partial \pi_3}{\partial y_3} + \frac{\pi_3}{\sqrt{2}(1+\beta)} = 0 \tag{7.10b}$$

$$BC(y_3 = -\sqrt{\frac{1+\beta}{1-\beta}}x_3) : \frac{(2\beta+1)}{\sqrt{1+\beta}}\frac{\partial \pi_3}{\partial x_3} - \frac{(2\beta-1)}{\sqrt{1-\beta}}\frac{\partial \pi_3}{\partial y_3} + \frac{\pi_3}{\sqrt{2}(1+\beta)} = 0. \quad (7.10c)$$

(4): The next transformation $T_3 : (x_3, y_3) \to (x_4, y_4)$ is given by

$$\begin{pmatrix} x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2(1+\beta)}} & 0 \\ 0 & \frac{1}{\sqrt{2(1+\beta)}} \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \quad (x_3, y_3) \in I\!R^2_\beta \quad (7.11)$$

Denoting $\pi_3(T_3(x_3, y_3))$ by $\pi_4(x_4, y_4)$, we obtain

$$\nabla^2 \pi_4 = \pi_4, \quad (x_4, y_4) \in I\!R^2_\beta \quad (7.12a)$$

$$BC(y_4 = \sqrt{\frac{1+\beta}{1-\beta}}x_4) : (2\beta+1)\frac{\partial \pi_4}{\partial x_4} + \sqrt{\frac{1+\beta}{1-\beta}}(2\beta-1)\frac{\partial \pi_4}{\partial y_4} + \pi_4 = 0 \quad (7.12b)$$

$$BC(y_4 = -\sqrt{\frac{1+\beta}{1-\beta}}x_4) : (2\beta+1)\frac{\partial \pi_4}{\partial x_4} - \sqrt{\frac{1+\beta}{1-\beta}}(2\beta-1)\frac{\partial \pi_4}{\partial y_4} + \pi_4 = 0. \quad (7.12c)$$

(5): Finally we recast this equation into polar co-ordinates with the the transformation $T_4 : (x_4, y_4) \to (r, \theta)$ given by

$$x_4 = r\cos\theta \quad \text{and} \quad y_4 = r\sin\theta$$

We retain the notation $I\!R^2_\beta$ for the region in the (r, θ) plane that is bounded by the straight lines $\theta = \tan^{-1}\sqrt{\frac{1+\beta}{1-\beta}}$ and $\theta = -\tan^{-1}\sqrt{\frac{1+\beta}{1-\beta}}$. Denoting $\pi_4(T(x_4, y_4)) = \phi(r, \theta)$, we finally obtain

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \phi, \quad (r, \theta) \in I\!R^2_\beta \quad (7.13a)$$

$$BC(\theta = \tan^{-1}\sqrt{\frac{1+\beta}{1-\beta}}) : \frac{2\beta}{\sqrt{1-\beta}}\frac{\partial \phi}{\partial r} - 2\frac{\sqrt{1+\beta}}{r}\frac{\partial \phi}{\partial \theta} + \sqrt{2}\phi = 0 \quad (7.13b)$$

$$BC(\theta = -\tan^{-1}\sqrt{\frac{1+\beta}{1-\beta}}) : \frac{2\beta}{\sqrt{1-\beta}}\frac{\partial \phi}{\partial r} + 2\frac{\sqrt{1+\beta}}{r}\frac{\partial \phi}{\partial \theta} + \sqrt{2}\phi = 0. \quad (7.13c)$$

We shall find a solution to this equation in the case when $\beta = \frac{1}{2}$. In this case the equation becomes,

$$\nabla^2 \phi = \phi, \quad (r, \theta) \in I\!R^2_{0.5} \tag{7.14a}$$

$$BC(\theta = \frac{\pi}{3}) : \frac{\partial \phi}{\partial r} - \frac{\sqrt{3}}{r} \frac{\partial \phi}{\partial \theta} + \phi = 0 \tag{7.14b}$$

$$BC(\theta = -\frac{\pi}{3}) : \frac{\partial \phi}{\partial r} + \frac{\sqrt{3}}{r} \frac{\partial \phi}{\partial \theta} + \phi = 0. \tag{7.14c}$$

The case $\beta = \frac{1}{2}$ is of importance because it corresponds to the situation when $\sigma_0 = \sigma$, i.e., when the inter-arrival and service distributions have the same limiting variance. This will always be the case if the service and inter–arrival time distributions are taken from the same family. For example consider the case of a Fork–Join queue with exponential inter–arrival and service distributions with rate $\frac{1}{\lambda}$ an $\frac{1}{\mu}$ respectively. Then as $\lambda \uparrow \mu$ in heavy traffic,

$$\sigma_0^2 = \frac{1}{\lambda^2} \to \frac{1}{\mu^2} = \sigma^2.$$

Equations similar to (7.14) have been encountered earlier by Harrison [10] and Foschini [8] in the context of the diffusion limit for queues in tandem. Guided by their work, we try a solution of the form

$$\phi(r, \theta) = \frac{1}{\sqrt{r}} e^{-r} \cos(\frac{\theta}{2}), \quad (r, \theta) \in I\!R^2_{0.5}. \tag{7.15}$$

Note that it satisfies the PDE as well as the boundary conditions (7.14b)–(7.14c), as can be verified by a direct substitution.

Our next objective is to obtain an expression for the density in terms of (x, y). Note that the transformation $T : (r, \theta) \to (x, y)$, which is a composition of the transformations $T = T_0^{-1} T_1^{-1} T_2^{-1} T_3^{-1} T_4^{-1}$, can be written as

$$\begin{aligned} ax &= (1 + \beta) r \cos\theta - \sqrt{1 - \beta^2} r \sin\theta \\ ay &= (1 + \beta) r \cos\theta + \sqrt{1 - \beta^2} r \sin\theta, \quad (r, \theta) \in I\!R^2_{0.5}. \end{aligned} \tag{7.16}$$

If we undo the transformation which corresponded to a multiplication by $e^{-\frac{1}{3}(x_1+y_1)}$, we obtain the function $\psi(r,\theta)$, where

$$\psi(r,\theta) = \frac{1}{\sqrt{r}} e^{-r(1+\cos\theta)} \cos(\frac{\theta}{2}), \quad (r,\theta) \in \mathbb{R}^2_{0.5}. \tag{7.17}$$

Letting $\psi(T(r,\theta)) = \varphi(x,y)$, the final solution is of the form $K\varphi(x,y)$ where the constant K is chosen so that

$$\int_0^\infty \int_0^\infty K\varphi(x,y)dxdy = 1. \tag{7.18}$$

We shall evaluate this integral on the (r,θ) plane where the calculations are much easier. It can easily be checked that the Jacobian J for the transformation (7.16) is given by $J = \frac{2r}{a^2}(1+\beta)\sqrt{1-\beta^2} = \frac{3r}{2a^2}\sqrt{3}$. The integral (7.14) then transforms to

$$3\sqrt{3}K \int_{\theta=0}^{\frac{\pi}{3}} \int_{r=0}^{\infty} \sqrt{r} e^{-r(1+\cos\theta)} \cos(\frac{\theta}{2}) dr d\theta = a^2. \tag{7.19}$$

Making the substitution $\gamma(\theta) = 1 + \cos\theta$, it follows that

$$\begin{aligned}
\int_0^\infty \sqrt{r} e^{-\gamma(\theta) r} dr &= \gamma^{-\frac{3}{2}}(\theta) \int_0^\infty \sqrt{u} e^{-u} du \\
&= \gamma^{-\frac{3}{2}}(\theta)\Gamma(\frac{3}{2}) \\
&= \frac{1}{4}\sqrt{\frac{\pi}{2}} \cos^{-3}\frac{\theta}{2}
\end{aligned}$$

Substituting this back into (7.19), it follows that

$$\begin{aligned}
a^2 &= \frac{3\sqrt{3}}{4}\sqrt{\frac{\pi}{2}} K \int_{\theta=0}^{\frac{\pi}{3}} \sec^{-2}(\frac{\theta}{2}) d\theta \\
&= \frac{3\sqrt{3}}{4}\sqrt{\frac{\pi}{2}} K [\tan(\frac{\theta}{2})]_0^{\frac{\pi}{3}} \\
&= \frac{3K}{2}\sqrt{\frac{\pi}{2}}
\end{aligned}$$

so that

$$K = \frac{2a^2}{3}\sqrt{\frac{2}{\pi}}$$

Hence the final solution is

$$\pi(x,y) = \frac{2a^2}{3}\sqrt{\frac{2}{\pi}}\varphi(x,y), \quad (x,y) \in I\!R_+^2. \tag{7.20}$$

Making use of the fact that

$$r = \frac{2a}{3}\sqrt{x^2 - xy + y^2}, \quad \cos\theta = \frac{x+y}{2\sqrt{x^2 - xy + y^2}} \tag{7.21}$$

and substituting for r and $\cos\theta$ in (7.17), we finally conclude to the following result.

Theorem 7.1 *The stationary density $\pi(x,y)$ of the diffusion for the waiting times in a symmetric two dimensional Fork–Join queue in heavy traffic, which satisfies $\sigma = \sigma_0$ is given by*

$$\pi(x,y) = a\sqrt{\frac{a}{3}}\frac{\sqrt{2\sqrt{x^2 - xy + y^2} + x + y}e^{-\frac{2a}{3}\sqrt{x^2-xy+y^2} - \frac{a}{3}(x+y)}}{\sqrt{x^2 - xy + y^2}},$$

$$(x,y) \in I\!R_+^2 \tag{7.22}$$

where $a = \frac{c}{\sigma^2}$.

Knessel [19] has also considered the problem of solving an equation similar to (7.3) from the point of view of the theory of singular perturbations, and obtained an expression for $\pi(x,y)$ in the case when x and y are very large. As expected, our solution (7.22) agrees with his for large x, y.

7.4 Calculations of the moments of the end–to–end delay

In this sub–section our objective is to obtain some information regarding the stationary density and the moments of the diffusion for the end–to–end delay of the Fork–Join queue.

Our first objective is to find an expression for the density of the equilibrium response time κ_∞, where

$$\kappa_\infty = \max\{\eta^1_\infty, \eta^2_\infty\}.$$

It is clear that

$$F(z) = I\!P(\kappa_\infty \leq z) = I\!P(\eta^1_\infty \leq z, \eta^2_\infty \leq z) = \int_0^z \int_0^z \pi(x,y)dxdy, \quad z \geq 0. \qquad (7.23)$$

We make a change of co-ordinates from (x,y) to (r,θ), by using the transformation (7.16). The square $[0,z] \times [0,z]$ in the (x,y) plane maps into the rhombus with sides of length $\frac{2az}{3}$ and vertices at $(0,0), (\frac{2az}{3}, \frac{\pi}{3}), (\frac{2az}{3}, 0)$ and $(\frac{2az}{3}, -\frac{\pi}{3})$ in the (r,θ) plane. Using the law of sines for triangles, it is clear that the limits of integration for r are from $r=0$ to $r = \frac{2az}{3}\frac{\sin\frac{\pi}{3}}{\sin(\frac{2\pi}{3}-\theta)} = \frac{az}{\sqrt{3}}\frac{1}{\sin(\frac{2\pi}{3}-\theta)}$. Hence

$$F(z) = 2\sqrt{\frac{6}{\pi}} \int_{\theta=0}^{\frac{\pi}{3}} \int_{r=0}^{\frac{az}{\sqrt{3}}\frac{1}{\sin(\frac{2\pi}{3}-\theta)}} \sqrt{r}e^{-r(1+\cos\theta)} \cos\frac{\theta}{2} dr d\theta, \quad z \geq 0. \qquad (7.24)$$

As before, let $\gamma(\theta) = 1 + \cos\theta$, so that with the substitution $u = \gamma(\theta)r$ we obtain

$$\int_{r=0}^{\frac{az}{\sqrt{3}}\frac{1}{\sin(\frac{2\pi}{3}-\theta)}} \sqrt{r}e^{-\gamma(\theta)r} dr = \gamma^{-\frac{3}{2}}(\theta) \int_{u=0}^{\frac{az\gamma(\theta)}{\sqrt{3}}\frac{1}{\sin(\frac{2\pi}{3}-\theta)}} \sqrt{u}e^{-u} du.$$

The resultant integral above is known as the incomplete Gamma function and occurs frequently in analysis. It is well known that

$$\int_0^x \sqrt{t}e^{-t}dt + \int_x^\infty \sqrt{t}e^{-t}dt = \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}. \qquad (7.25)$$

Hence (7.24) can be re-written as

$$\begin{aligned} F(z) &= 2\sqrt{\frac{6}{\pi}} \int_{\theta=0}^{\frac{\pi}{3}} \cos\frac{\theta}{2}\gamma^{-\frac{3}{2}}(\theta)[\frac{\sqrt{\pi}}{2} - \int_{\frac{az\gamma(\theta)}{\sqrt{3}}\frac{1}{\sin(\frac{2\pi}{3}-\theta)}}^\infty \sqrt{u}e^{-u}du]d\theta \\ &= 1 - 2\sqrt{\frac{6}{\pi}} \int_{\theta=0}^{\frac{\pi}{3}} \int_{u=\frac{az\gamma(\theta)}{\sqrt{3}}\frac{1}{\sin(\frac{2\pi}{3}-\theta)}}^\infty \gamma^{-\frac{3}{2}}(\theta)\cos\frac{\theta}{2}\sqrt{u}e^{-u}dud\theta \\ &= 1 - \sqrt{\frac{3}{\pi}} \int_{\theta=0}^{\frac{\pi}{3}} \int_{u=\frac{2az}{\sqrt{3}}\cos^2\frac{\theta}{2}\frac{1}{\sin(\frac{2\pi}{3}-\theta)}}^\infty \frac{\sqrt{u}e^{-u}}{\cos^2\frac{\theta}{2}} dud\theta, \quad z \geq 0. \end{aligned} \qquad (7.26)$$

Using (7.26), the density function $f(z)$ of the response time is given by

$$f(z) = \frac{dF(z)}{dz} = \sqrt{\frac{3}{\pi}} \int_{\theta=0}^{\frac{\pi}{3}} \frac{1}{\cos\frac{\theta}{2}} \delta(\theta) \sqrt{\delta(\theta) z} e^{-\delta(\theta) z} d\theta, \quad z \geq 0 \tag{7.27}$$

where $\delta(\theta) = \frac{2a}{\sqrt{3}} \frac{\cos^2\frac{\theta}{2}}{\sin(\frac{2\pi}{3}-\theta)}$.

Using this expression it is possible to obtain a formula for the n^{th} moment of the response time, as is done next. Note that

$$\begin{aligned} \mathbb{E}\kappa_\infty^n &= \int_{z=0}^{\infty} z^n f(z) dz \\ &= \sqrt{\frac{3}{\pi}} \int_{\theta=0}^{\frac{\pi}{3}} \delta(\theta)\sqrt{\delta(\theta)} \frac{d\theta}{\cos^2\frac{\theta}{2}} \int_{z=0}^{\infty} z^{n+\frac{1}{2}} e^{-\delta(\theta) z} dz \\ &= \Gamma(n+\frac{3}{2}) \sqrt{\frac{3}{\pi}} \int_{\theta=0}^{\frac{\pi}{3}} \frac{1}{\cos^2\frac{\theta}{2}} \frac{1}{\delta^n(\theta)} d\theta. \end{aligned} \tag{7.28}$$

Substituting for $\delta(\theta)$ and a, we finally obtain the following result.

Proposition 7.1 *The n^{th} moment of the stationary density of the diffusion for the end–to–end delay in a symmetrical two dimensional Fork–Join queue in heavy traffic which satisfies $\sigma = \sigma_0$, is given by*

$$\mathbb{E}\kappa_\infty^n = \Gamma(n+\frac{3}{2}) \sqrt{\frac{3}{\pi}} (\frac{\sqrt{3}}{2} \frac{\sigma^2}{c})^n P_n \tag{7.29}$$

where

$$\Gamma(n+\frac{3}{2}) = \frac{(2n+1)(2n-1)\ldots 3.1}{2^{n+1}} \sqrt{\pi} \tag{7.30}$$

and

$$P_n = \int_{\theta=0}^{\frac{\pi}{3}} \frac{\sin^n(\frac{2\pi}{3}-\theta)}{\cos^{2n+2}\frac{\theta}{2}} d\theta. \tag{7.31}$$

~~The reader may check that for the case $n = 1$, this formula agrees with the expression for $E\kappa_\infty$ derived earlier.~~ The integral (7.31) is evaluated for some values of n in Section 7.5.

Equation (7.29) suggests the following formula for the heavy traffic limit of the n^{th} moment of the response time: Consider a symmetric Fork–Join queue governed by an arrival process with mean $\frac{1}{\lambda}$ and variance $\sigma_0^2(\lambda)$, and a service time distribution with mean $\frac{1}{\mu}$ and variance σ^2. Further assume that $\lim_{\lambda\uparrow\mu}\sigma_0^2(\lambda) = \sigma^2$. The heavy traffic limit for the n^{th} moment of the end–to–end delay of this queue $\overline{T}^n(\lambda)$, is given by

$$\lim_{\lambda\uparrow\mu}(\mu-\lambda)^n\overline{T}^{(n)}(\lambda) = \Gamma(n+\frac{3}{2})\sqrt{\frac{3}{\pi}}(\frac{\sqrt{3}}{2}\sigma^2\mu^2)^n P_n \tag{7.32}$$

where $\Gamma(n+\frac{3}{2})$ and P_n are defined in (7.30)-(7.31).

We now provide a formula for the normalized correlation between the delay processes of the two queues in heavy traffic. This is given by

$$\mathbb{E}(\eta_\infty^1\eta_\infty^2) = \int_0^\infty\int_0^\infty xy\pi(x,y)dxdy.$$

Making the usual change of co–ordinates from (x,y) to (r,θ), we obtain after some calculations that

$$\mathbb{E}(\eta_\infty^1\eta_\infty^2) = \frac{11}{8}(\frac{\sigma^2}{c})^2. \tag{7.33}$$

Note that the two queues by themselves behave like $GI/GI/1$ queues, so that

$$\mathbb{P}(\eta_\infty^k \le x) = 1 - e^{-\frac{c}{\sigma^2}x}, \quad k=1,2$$

and

$$\mathbb{E}\eta_\infty^k = \frac{\sigma^2}{c}, \quad k=1,2.$$

It follows that

$$Cov(\eta_\infty^1\eta_\infty^2) = \mathbb{E}(\eta_\infty^1\eta_\infty^2) - \mathbb{E}\eta_\infty^1\mathbb{E}\eta_\infty^2 = \frac{3}{8}(\frac{\sigma^2}{c})^2. \tag{7.34}$$

This implies the following result.

Proposition 7.2 *Consider a symmetric K–dimensional Fork–Join queue governed by an arrival process with mean $\frac{1}{\lambda}$ and variance $\sigma_0^2(\lambda)$, and a service time distribution with mean*

$\frac{1}{\mu}$ *and variance* σ^2. *Further assume that* $\lim_{\lambda\uparrow\mu}\sigma_0^2(\lambda) = \sigma^2$. *If* $(W^1(\lambda),\ldots,W^K(\lambda))$ *represents the steady state vector of queueing delays in the system, then*

$$\lim_{\lambda\uparrow\mu} Corr(W^i(\lambda)W^j(\lambda)) = \frac{3}{8}, \quad 1 \le i,j \le K, \quad i \ne j. \tag{7.35}$$

It is a remarkable coincidence that the asymptotic correlation $\frac{3}{8}$, almost equals the constant $\frac{11}{4}$ that was obtained by Nelson and Tantawi [22] as part of their heuristic approximation. We also note that the correlation between two queues in the system is crucially dependent on the parameter β. We just showed that for the case $\beta = \frac{1}{2}$, the coefficient of variation is given by $\frac{3}{8}$. For the case $\beta = 0$, it is given by zero, since in this case the two queues are independent, while in the case $\beta = 1$ it is given by one, since in this case both queues are perfectly synchronized with one another. Hence we observe that as the service times become more deterministic, i.e., as β increases, the two queues become more correlated with one another.

7.5 Tables for P_n

The co-efficient P_n defined in (7.31) have been calculated for $n = 1,\ldots,4$, with the help of the symbolic computation language MACSYMA, and set down in the table below.

n	P_n	$\mathbb{E}\kappa_\infty^n$
1	$\frac{11}{9}$	$\frac{11}{8a}$
2	$\frac{3}{5}\sqrt{3}$	$\frac{81}{32a^2}$
3	$\frac{1759}{1260}$	$\frac{10.3}{a^3}$
4	$\frac{59123}{68040}\sqrt{3}$	$\frac{43.3}{a^3}$

8. An heuristic formula for the heavy traffic limit for general values of β

Recall that the parameter β was defined by

$$\beta = \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2}$$

where σ_0 is the variance of the inter–arrival times and σ is the variance of the service times. The heavy traffic limit (7.32) when $K = 2$ was obtained for special case $\beta = \frac{1}{2}$. In this section we propose an heuristic approximation for the heavy traffic limit, for general values of β in $[0, 1]$. We do so by taking advantage of the fact that the Fork–Join queue is easy to analyze for $\beta = 0$ and $\beta = 1$. In particular we note that,

(1): $\beta = 0$: In this case the two queues are decoupled from each other since the arrival stream is deterministic, and using (6.31), we can write

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)^n \overline{T}^{(n)}(\lambda) = n!(\frac{\sigma^2 \mu^2}{2})^n (2 - \frac{1}{2^n}) \qquad n = 1, 2 \ldots (8.1)$$

(2): $\beta = \frac{1}{2}$: In this case $\sigma_0 = \sigma$ and according to (7.32),

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)^n \overline{T}(n)(\lambda) = \Gamma(n + \frac{3}{2}) \sqrt{\frac{3}{\pi}} (\frac{\sqrt{3}}{2} \sigma^2 \mu^2)^n P_n \qquad n = 1, 2 \ldots (8.2)$$

(3): $\beta = 1$: In this case $\sigma = 0$ and the system behaves like a $GI/D/1$ queue so that

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda) \overline{T}_K(\lambda) = n!(\frac{\sigma_0^2 \mu^2}{2})^n \qquad n = 1, 2 \ldots (8.3)$$

Observing the structure of (8.1)–(8.3), we may venture to write down the following expression for the heavy traffic limit for a general value of β.

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda) \overline{T}^{(n)}(\lambda) = M_n(\beta) \left[\frac{\sigma^2 + \sigma_0^2}{2} \mu^2 \right]^n, \quad 0 \leq \beta \leq 1, \qquad n = 1, 2 \ldots (8.4)$$

where for $n = 1, 2 \ldots$,

$$M_n(0) = n!(2 - \frac{1}{2^n}) \qquad (8.5a)$$

$$M_n(\frac{1}{2}) = \Gamma(n + \frac{3}{2}) \sqrt{\frac{3}{\pi}} (\frac{\sqrt{3}}{2})^n P_n \qquad (8.5b)$$

$$M_n(1) = n! \qquad (8.5c)$$

In the absence of any further information we may use a quadratic approximation for $M_n(\beta), 0 \leq \beta \leq 1$. This leads to the following formula.

$$\lim_{\lambda \uparrow \mu}(\mu-\lambda)^n\overline{T}^{(n)}(\lambda) = (n!(2-\frac{1}{2^n}) + \left[4\Gamma(n+\frac{3}{2})\sqrt{\frac{3}{\pi}}(\frac{\sqrt{3}}{2})^n P_n - n!(7-\frac{3}{2^n})\right]\beta + \left[2n!(3-\frac{1}{2^n}) - 4\Gamma(n+\frac{3}{2})\sqrt{\frac{3}{\pi}}(\frac{\sqrt{3}}{2})^n P_n\right]\beta^2)\left[\frac{\sigma^2+\sigma_0^2}{2}\mu^2\right]^n$$

$$0 \leq \beta \leq 1, \quad n = 1, 2 \ldots (8.6)$$

heavy

We combine the heavx traffic limit (8.6) with light traffic limits in [29] to obtain interpolation approximations for all values of the traffic intensity.

APPENDIX A

We first give a definition of convex–increasing and strong stochastic orderings for continuous time stochastic processes.

Definition A1. *Let X and Y be two real-valued RVs. The RV X is said to be smaller than the RV Y in the sense of strong stochastic ordering if*

$$\mathbb{E}f(X) \leq \mathbb{E}f(Y)$$

for all non–decreasing functions $f : \mathbb{R} \to \mathbb{R}$. This is denoted as $X \leq_{st} Y$.

The RV X is smaller than the RV Y in the sense of convex increasing stochastic ordering if

$$\mathbb{E}f(X) \leq \mathbb{E}f(Y)$$

for all convex non–decreasing functions $f : \mathbb{R} \to \mathbb{R}$. This is denoted as $X \leq_{icx} Y$.

Let the symbol $\prec$ denote one of the stochastic orderings $\leq_{st}$ or $\leq_{icx}$. Let $X \equiv \{X_t, t \geq 0\}$ and $Y \equiv \{Y_t, t \geq 0\}$ be two real-valued stochastic processes. The process X is smaller than Y with respect to $\prec$, denoted as $X \prec Y$, if

$$X_t \prec Y_t, \quad t \geq 0.$$

We now introduce the concept of associated stochastic processes.

Definition A2. *The real-valued RVs $\{X^1, \ldots, X^K\}$, are associated if and only if, the inequality*

$$\mathbb{E}[f(X)h(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[h(X)]$$

holds for all pairs of monotone non-decreasing mappings $f, h : \mathbb{R}^K \to \mathbb{R}$ for which these expectations exist.

The real-valued stochastic processes $X^k \equiv \{X_t^k, t \geq 0\}, 1 \leq k \leq K$, are associated if and only if, for all $t \geq 0$, the RVs $\{X_t^1, \ldots, X_t^K\}$ are associated.

Definition A3 *The stochastic processes $\overline{X} \equiv \{\overline{X}_t^k, t \geq 0\}, 1 \leq k \leq K$, are said to form independent versions of the stochastic processes $X \equiv \{X_t^k, t \geq 0\}, 1 \leq k \leq K$, if*

(i): *For all $t \geq 0$, the RVs $\{\overline{X}_t^1, \ldots, \overline{X}_t^K\}$ are mutually independent, and*

(ii): *For every $1 \leq k \leq K$ and $t \geq 0$, the RVs X_t^k and $\overline{X}_t^k$ have the same probability distribution.*

The following result [BarPr] is an immediate consequence of this definition.

Lemma A1. *If the stochastic processes $X \equiv \{X_t^k, t \geq 0\}, 1 \leq k \leq K$, are associated, then the inequality*

$$\max_{1 \leq k \leq K} X_t^k \leq_{st} \max_{1 \leq k \leq K} \overline{X}_t^k \quad t \geq 0$$

holds true.

The following lemma [5] is very useful.

Lemma A2.

(i) : Independent RVs are always associated.

(ii) : The union of independent collections of associated RVs forms a set of associated RVs.

(iii) : Any subset of a family of associated RVs forms a set of associated RVs.

(iv) : A monotone non-decreasing function of associated RVs generates a set of associated RVs.

REFERENCES

[1] F. Baccelli,"Two parallel queues created by arrivals with two demands," *INRIA Rapport de Recherche*, No. 426, (1985).

[2] F. Baccelli, A.M. Makowski and A. Shwartz, "Fork-Join queue and related systems with synchronization constraints: Stochastic ordering, approximations and computable bounds," Electrical Engineering Technical Report, No. TR-87-01 University of Maryland, College Park (1987).

[3] F. Baccelli, W.A. Massey and D. Towsley, "Acyclic fork-join queueing networks," *INRIA Rapport de Recherche*, No. 688 (1987).

[4] F. Baccelli and A.M. Makowski,"Queueing systems with synchronization constraints," *Proceeding of the IEEE*, (1989).

[5] R.E. Barlow and F. Proschan, *Statistical theory of reliability and life testing*, Holt, Rinehart and Winston, Reading, MA.

[6] P. Billingsley, *"Convergence of probability measures,"* J. Wiley and Sons, New York, (1968).

[7] L. Flatto and S. Hahn,"Two parallel queues created by arrivals with two demands I," *SIAM J. Appl. Math.*, Vol. 44, pp. 1041-1053 (1984).

[8] G.J. Foschini, "Equilibra for diffusion models of pairs of communicating computers-Symmetric case," *IEEE Trans. on IT*, Vol. 28, No. 2, pp. 273-284 (1982).

[9] J.M. Harrison, "The heavy traffic approximation for single server queues in series," *J. Appl. Prob.*, No. 10, pp. 613-629 (1973).

[10] J.M. Harrison,"The diffusion approximation for tandem queues in heavy traffic," *Adv. Appl. Prob.*, No. 10, pp. 886-905 (1978).

[11] J.M. Harrison and M.I. Reiman,"On the distribution of multidimensional reflected Brownian motion," *SIAM J. Appl. Math.*, Vol. 41, No. 2, pp. 345-361 (1981).

[12] J.M. Harrison and M.I. Reiman, "Reflected Brownian motion on an orthant," *Ann. Prob.*, Vol. 9, No. 2, pp. 302-308 (1981).

[13] J.M. Harrison, *Brownian motion and stochastic flow systems*, J. Wiley and Sons, New York (1985).

[14] D.L. Iglehart and W. Whitt, "Multiple channel queues in heavy traffic. I," *Adv. Appl. Prob.*, No. 2, pp. 150-177 (1970).

[15] D.L. Iglehart and W. Whitt, "Multiple channel queues in heavy traffic. II: Sequences, networks and batches," *Adv. Appl. Prob.*, No. 2, pp.355-369 (1970).

[16] D.L. Iglehart, "Weak convergence in queueing theory," *Adv. Appl. Prob.*, Vol. 5, pp. 570-594 (1973).

[17] J.F.C. Kingman, "The single server queue in heavy traffic," *Proc. Cambridge Philos. Soc.* No. 57, pp. 902-904 (1961).

[18] J.F.C. Kingman, "On queues in heavy traffic," *J. Roy. Statist. Soc. B.* Vol. 25, pp. 383-392 (1962). J. Wiley and Sons, New York (1975).

[19] C. Knessl, "On the diffusion approximation to a fork and join queueing model," *Technical Report No. AM88-02*, Dept. of Mathematics, Statistics and Computer Science, University of Chicago at Illinois (1988).

[20] D.V. Lindley, "The theory of queues with a single server," *J. Appl. Prob.*, Vol. 10, pp. 109-121 (1973).

[21] T. Lindvall, "Weak convergence of probability measures and random functions in the function space $D[0,\infty)$," *J. Appl. Prob.*, Vol. 10, pp. 109-121 (1973).

[22] R. Nelson and A.N. Tantawi, "Approximate analysis of fork-join synchronization in parallel queues," *IEEE Trans. on Computers*, Vol. 37, No. 6, pp. 739-743 (1988).

[23] K.R. Parthasarthy, *Probability measures on metric spaces*, Academic Press, New York (1967).

[24] Y. Prohorov, "Convergence of random processes and limit theorems of probability theory," *Theor. Probability Appl.*, Vol. 1, pp. 157-214 (1956).

[25] Y. Prohorov, "Transient phenomena in processes of mass service," *Litovsk. Mat. Sb.*, Vol. 3, pp. 199-205 (1963).

[26] M.I. Reiman, "Open queueing networks in heavy traffic," *Maths. of Oper. Res.*, Vol. 9, No. 3, pp. 441-458 (1984).

[27] D. Stoyan, *Comparison methods for queues and other stochastic models*, English Translation (D.J. Daley, Editor), J. Wiley and Sons, New York, (1984).

[28] S. Varma, "Heavy and light traffic approximations for queues with synchronization

constraints," PhD Thesis, University of Maryland, (1990).

[29] S. Varma and A.M. Makowski, "Interpolation approximations for Fork–Join queues," In preparation, (1990).

[30] S. Varma and A.M. Makowski, "Approximations for homogeneous acyclic Fork–Join networks," In preparation, (1990).

[31] A.M. Makowski, T.K. Philips and S. Varma,"Comparison of scheduling strategies in multiprocessor systems," In preparation (1990).

[32] S. Varma and A.M. Makowski, "Approximations for synchronized networks," In preparation, (1990).

[33] W.Whitt,"Weak convergence theorems for queues in heavy traffic," Ph.D. Thesis, Cornell University (1968).

[34] W. Whitt,"Weak convergence of probability measures on the function space $C[0,\infty)$." *Ann. Math. Stat.*, Vol. 41, No. 3, pp. 939-944 (1970).

[35] W. Whitt,"Heavy traffic limit theorems for queues: A survey," Lecture Notes in Economics and Mathematical Systems, No. 98, Springer-Verlag, Berlin, pp. 307-350 (1974).

[36] W. Whitt,"Some useful functions for functional limit theorems," *Maths. of Oper. Res*, Vol. 5, No. 1, pp. 67-85 (1980).

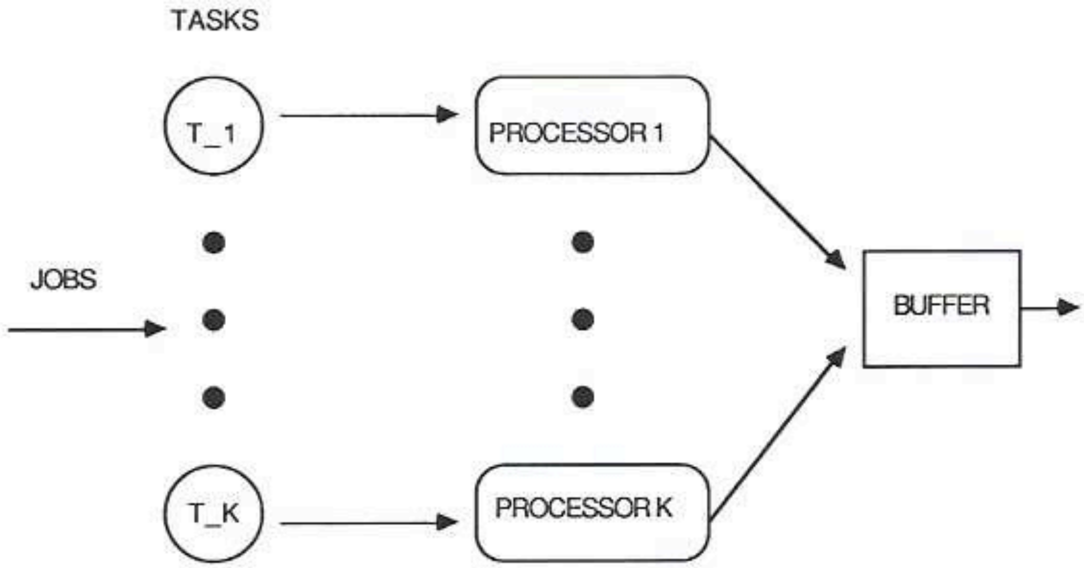

Fig. 1. A parallel processing system

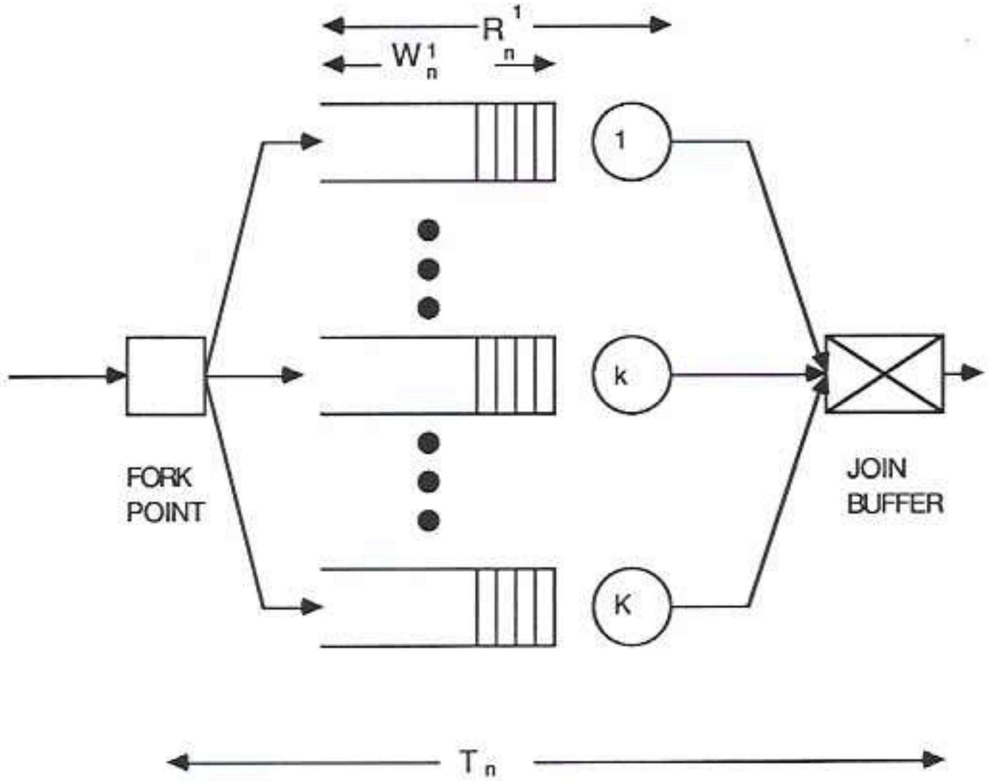

Fig. 2. A fork-join queue