APPROXIMATIONS FOR ACYCLIC FORK–JOIN NETWORKS

by

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ABSTRACT

Acyclic fork–join networks constitute a class of queueing models that exhibit synchronization delays in addition to queueing delays in their behavior. They are important in the analysis of multi–processor systems serving jobs with precedence constraints among tasks. Our objective is to obtain approximations for the average response time in these networks. We first obtain the heavy traffic diffusion limit for the queue delay processes. However this limiting diffusion has a complicated form and it is difficult to obtain its stationary distribution. Basing ourselves on the class of approximations that were given for the special case of parallel fork–join queues [24], we propose a family of approximations for acyclic fork–join networks. These approximations are validated by applying them to a particular acyclic fork–join network, and good agreement with simulations is observed.

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1. Introduction

Fork–join systems arise as models for parallel processing systems. For example, consider a parallel processing system with K processors which is subject to an arrival stream of jobs consisting of multiple tasks (Fig 1). These tasks have precedence constraints among themselves, which is represented by the directed edges of an acyclic graph. There are two kinds of nodes that characterize such a graph, i.e., fork nodes and join nodes. A fork node has more that one arrow emanating out of it and corresponds to the fact that multiple tasks may commence execution, once the fork–node task has finished execution. A join node has more that one arrow incident upon it, which corresponds to the fact that the join node task may not begin execution until certain tasks in the job have finished their execution. A fundamental problem in the performance evaluation of parallel processing systems is to obtain the execution time statistics of such jobs in the system.

If we assume that the number of processors is equal to the number of tasks, all the jobs have the same task graph and all inter–processor communication delays can be ignored, then the system can be modeled by an acyclic fork–join queueing network [2] (Fig 2). (Recently a model also has been proposed that also incorporates communication delays [26]). The problem of interest is to obtain the statistics of the response time of a customer in these networks. In general this is a very difficult problem, because the presence of synchronization delays (in addition to queueing delays) destroys the product form property which characterizes Jackson networks.

In this paper our objective is to obtain approximations for homogeneous, acyclic fork–join networks. In Section 2 we demonstrate the convergence to a diffusion process for the various queueing delays in the network, so that the problem of obtaining heavy traffic limits is converted to the problem of obtaining the stationary distribution for this diffusion. However, the limiting diffusion has an extremely complicated form and it seems difficult to obtain its stationary distribution. Hence, basing ourselves on the experience gained in analyzing parallel fork–join systems [24] we propose some heuristic approximations for homogeneous networks. These approximations agree extremely well with simulation results.

2. Heavy traffic limit for acyclic fork–join networks

2.1 The model

Baccelli, Massey and Towsley [2] extended the notion of a single stage fork–join queue to acyclic fork–join networks. The single stage fork–join queue analyzed in Section 2.2 is a special case of these networks. To see the motivation for introducing these more general networks, the reader may consult the survey paper [3]. We now introduce the notation and definitions associated with this network, most of which is borrowed from [3].

The acyclic fork–join network under consideration is represented by an acyclic graph $G = (V, E)$ where V is a set of B FIFO queues labeled $i = 1, \ldots, B$ and E is a set of links such that (i, j) in E implies $j > i$. Also add for the sake of convenience fictitious queues 0 and $B + 1$, which act respectively as source and sink for the network.

For $1 \leq i \leq B$, we define the set of immediate predecessors $p(i)$ of queue i as the set of queues that have direct link to queue i, i.e.,

$$p(i) = \{j \in (1, \ldots, B) \mid (j, i) \in E\} \tag{2.1}$$

and the set of immediate successors $s(i)$ of queue i, as the set of queues to which i has a direct link, i.e.,

$$s(i) = \{j \in (1, \ldots, B) \mid (i, j) \in E\}. \tag{2.2}$$

We also denote as $s(0)$, the set of queues with no incoming links and as $p(B + 1)$, the set of queues with no outgoing links. It will be assumed that the numbering of the queues is such that

$$s(0) = \{1, \ldots, B'\}, \quad B' \leq B$$

and

$$p(B + 1) = \{B'', \ldots, B\}, \quad B'' \leq B.$$

We now describe the operation of the network. We assume that customers are being created at the source which acts as the outside world for the network. These exogenous customers enter the network through the queues in $s(0)$ and traverse it upon following certain synchronization rules $(SR_1) - (SR_3)$ described below. Finally customers leave the network from the queues in $p(B + 1)$ by being absorbed into the network sink and disappearing. We now specify the synchronization rules that govern the network.

(SR_1): The exogenous customers created at the source are routed instantaneously to the queues in $s(0)$ under the constraint of a Fork primitive, i.e., the n^{th} arrival date to each one of the queues in $s(0)$ coincides with the n^{th} date of customer creation. An alternate way of viewing this constraint is to assume that upon its creation, a customer creates B' replicas of itself which are then dispatched at the same time and instantaneously to the queues in $s(0)$, one replica per queue.

(SR_2): A service completion in some queue i in $s(0)$ will not systematically trigger an arrival to a queue in $s(i)$. In fact, more generally, the arrivals to queue j, with $B' < j \leq B$, are generated as follows: Assume the sequence of service completions to be known for all queues i, with $1 \leq i \leq j$ and $B' < j \leq B$. The n^{th} arrival date to queue j coincides with the latest date among all the n^{th} service completions at the queues in $p(j)$. Due to the acyclic structure of (V, E), this mechanism will successively define the arrival patterns to queues $B'+1, B'+2, \ldots, B$.

(SR_3): Customers leave the network through the queues in $p(B+1)$ in the form of a single output stream by imposing the following synchronization of the join type: The n^{th} network departure is defined as the latest date among the dates of n^{th} service completions in the queues $B'', B''+1, \ldots, B$.

2.2 Recursive representation of the delays

In this section a recursive representation for the delays in the network is provided. The material of this section is borrowed from [3].

Given an acyclic graph $G = (V, E)$, the performance measures associated with the corresponding network are fully specified by $B+1$ sequences of $\mathbb{R}_+$-valued RVs with the interpretation that for all $n = 0, 1 \ldots$, and $1 \leq j \leq B$,

τ_n : Arrival epoch of the n^{th} customer into the network.

v_n^j : Service time requirement of the n^{th} customer to be served in queue j.

We assume the system to be initially empty and adopt the convention that the 0^{th} exogenous customer is created at time $t = 0$, so that $\tau_0 = 0$. In terms of these RVs we define the following quantities for all $n = 0, 1 \ldots$ and all $1 \leq j \leq B$,

u_n : Inter-arrival time between the $(n+1)^{rst}$ and n^{th} exogenous customers ($= \tau_{n+1} -$

τ_n).

D_n^j : Delay between the arrival of the n^{th} exogenous customer in the network and the beginning of the n^{th} service in queue j.

W_n^j : Waiting time of the n^{th} exogenous customer in the buffer of queue j.

T_n : End-to-end delay or network response time of the n^{th} exogenous customer.

The following recursion between these variables was established by Baccelli, Massey and Towsley [2].

Lemma 2.1. *Consider the acyclic fork–join network defined above. If the system is initially empty, then for* $1 \leq j \leq B$, *the recursions*

$$D_0^j = \max_{i \in p(j)} \{D_0^i + v_0^i\}$$

$$D_{n+1}^j = \max\{\max_{i \in p(j)} \{D_{n+1}^i + v_{n+1}^i\}, D_n^j + v_n^j - u_{n+1}\} \quad n = 0, 1 \ldots (2.3)$$

and

$$W_0^j = 0$$

$$W_{n+1}^j = \max\{0, W_n^j + \max_{i \in p(j)} \{D_n^i + v_n^i\} - \max_{i \in p(j)} \{D_{n+1}^i + v_{n+1}^i\} + v_n^j - u_{n+1}\},$$

$$n = 0, 1 \ldots (2.4)$$

hold where the maximum over an empty set is zero by convention. Moreover the network response time of the n^{th} *customer is given by*

$$T_n = \max_{i \in p(B+1)} \{D_n^i + v_n^i\}. \qquad n = 0, 1 \ldots (2.5)$$

Proof. Since the system is initially empty, the boundary conditions (4.3)–(4.4) are immediate from the synchronization rules (SR_1)–(SR_2). Customers arriving to queue j in $s(0)$ do so according to the pattern of exogenous arrivals, so that D_n^j corresponds to the n^{th} waiting time in a FIFO queue generated by the sequences of interarrivals $\{u_{n+1}\}_0^\infty$ and service requirements $\{v_n^j\}_0^\infty, 1 \leq j \leq B'$. Writing the corresponding Lindley equation, we get

$$D_{n+1}^j = \max\{0, D_n^j + v_n^j - u_{n+1}\}, \quad 1 \leq j \leq B' \qquad n = 0, 1 \ldots (2.6)$$

and this reduces to (2.3), since $p(j) = \emptyset$ for j in $s(0)$.

For $B' < j \le B$, we fix $n = 0, 1 \ldots$. The $(n+1)^{rst}$ service completion at queue i in $p(j)$ takes place at time $\tau_{n+1} + D^i_{n+1} + v^i_{n+1}$, so that by applying the synchronization rule (SR_2), we see that the $(n+1)^{rst}$ arrival to queue j takes place at time $\tau_{n+1} + \max_{i \in p(j)}\{D^i_{n+1} + v^i_{n+1}\}$. Since the server at queue j becomes available for service at time $\tau_n + D^j_n + v^j_n$, we readily obtain (2.3).

In order to derive (2.4) we just have to note the relations

$$W^j_n = D^j_n - \max_{i \in p(j)} \{D^i_n + v^i_n\}, \quad 1 \le j \le B. \qquad n = 0, 1 \ldots$$

∎

We now state a result regarding the stability of these networks. First we make the assumption **(Ie)** where

(Ia): The sequences $\{u_{n+1}\}_0^\infty$ and $\{v^j_n\}_0^\infty, j = 1, \ldots, B$, are iid with finite second moments and mutually independent.

For $n = 0, 1 \ldots$, we set

$$u = I\!E(u_{n+1}) < \infty, \quad \sigma_0^2 = Var(u_{n+1}) < \infty$$

and

$$v^j = I\!E(v^j_n) < \infty, \quad \sigma_j^2 = Var(v^j_n) < \infty, \quad 1 \le j \le B$$

Again, as for the simple fork–join queue, we consider the system to be stable if the vector of delays $\{(D^1_n, \ldots, D^B_n)\}_0^\infty$ converges jointly in distribution as $n \uparrow \infty$ to a proper random vector $(D^1, \ldots, D^B)$. The stability conditions for this system were given in [2], and are reproduced below.

Lemma 2.2. *Assume that condition* **(Ie)** *holds. The system is stable iff*

$$v^j < u, \quad 1 \le j \le B. \qquad (2.7)$$

2.3 The diffusion limit

In the last section we saw that the acyclic fork–join network will be stable provided $v^j < u, 1 \le j \le B$. The system is said to be in heavy traffic if $v^j \approx u$ for at least one of the queues. In this section our objective is to develop heavy traffic diffusion limits for the delay processes in these networks. The methodology that we employ is the same as the one used in [10]. In short we shall use the recursions (2.3)–(2.4) to connect the delay processes to partial sums of iid RVs and then use well-known functional central limit theorems for these partial sums in order to deduce the corresponding limit theorems for the delay processes by means of the continuous mapping theorem.

We now consider a sequence of these networks indexed by $r = 1, 2 \ldots$, each of which satisfies condition **(Ia)**. Moreover assume that

(Ib): As $r \uparrow \infty$,

$$\sigma_j(r) \to \sigma_j, \quad 0 \le j \le B$$
$$[u(r) - v^j(r)]\sqrt{r} \to c_j. \quad 1 \le j \le B$$

(Ic): For some $\epsilon > 0$,

$$\sup_{r,j} \{\mathbb{E}\{\mid u_1(r) \mid^{2+\epsilon}\}, \mathbb{E}\{\mid v_1^j(r) \mid^{2+\epsilon}\}\} < \infty.$$

For $1 \le j \le B$ and $r = 1, 2 \ldots$, define the partial sums

$$V_0^j(r) = 0,$$
$$V_n^j(r) = v_0^j(r) + \ldots + v_{n-1}^j(r), \qquad n = 1, 2 \ldots (2.8a)$$

and

$$U_0(r) = 0,$$
$$U_n(r) = u_1(r) + \ldots + u_n(r). \qquad n = 1, 2 \ldots (2.8b)$$

For $r = 1, 2 \ldots$, define the stochastic processes $\xi^j(r) \equiv \{\xi_t^j(r), t \ge 0\}, 0 \le j \le B$, with sample paths in $D[0, \infty)$ by

$$\xi_t^0(r) = \frac{U_{[rt]}(r) - u(r)[rt]}{\sqrt{r}}, \quad t \ge 0 \qquad (2.9a)$$

and

$$\xi_t^j(r) = \frac{V_{[rt]}^j(r) - v^j(r)[rt]}{\sqrt{r}}, \quad 1 \le j \le B, \quad t \ge 0. \tag{2.9b}$$

Let $\xi^j \equiv \{\xi_t^j, t \ge 0\}, 0 \le j \le B$, be $B+1$ independent Wiener processes. Lemma 2.4.3 shows that the stochastic processes defined in (2.9) converge weakly to these Wiener processes.

Lemma 2.3 *As* $r \uparrow \infty$,

$$(\xi^0(r), \xi^1(r), \ldots, \xi^B(r)) \Rightarrow (\sigma_0\xi^0, \sigma_1\xi^1, \ldots, \sigma_B\xi^B) \tag{2.10}$$

in $D[0,\infty)^{B+1}$.

Proof. Equation (2.10) follows directly by Prohorov's functional central limit theorem for triangular arrays [16] under assumptions **(Ia)-(Ic)**. ∎

For $r = 1, 2 \ldots$, we set

$$S_0^j(r) = 0$$

$$S_n^j(r) = V_n^j(r) - U_n(r), \qquad n = 1, 2 \ldots \tag{2.11}$$

and define the stochastic processes $\zeta^j(r) \equiv \{\zeta_t^j(r), t \ge 0\}, 1 \le j \le B$, with sample paths in $D[0,\infty)$ by

$$\zeta_t^j(r) = \frac{S_{[rt]}^j(r)}{\sqrt{r}}, \quad 1 \le j \le B, \quad t \ge 0. \tag{2.12}$$

We also define the stochastic processes $\zeta^j \equiv \{\zeta_t^j, t \ge 0\}, 1 \le j \le B$, by

$$\zeta_t^j = \sigma_j\xi_t^j - \sigma_0\xi_t^0 - c_j t, \quad 1 \le j \le B, \quad t \ge 0. \tag{2.13}$$

The process $(\zeta^1, \ldots, \zeta^K)$ is a K–dimensional diffusion process with drift vector c and covariance matrix R given by

$$c = (-c_1, \ldots, -c_K)$$

and

$$R = \begin{pmatrix} \sigma_1^2 + \sigma_0^2 & \sigma_0^2 & \cdots & \sigma_0^2 \\ \sigma_0^2 & \sigma_2^2 + \sigma_0^2 & \cdots & \sigma_0^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_0^2 & \sigma_0^2 & \cdots & \sigma_K^2 + \sigma_0^2 \end{pmatrix}.$$

The next result shows that the stochastic process $(\zeta^1(r), \ldots, \zeta^B(r))$ converges weakly to $(\zeta^1, \ldots, \zeta^B)$.

Lemma 2.4 *As* $r \uparrow \infty$,

$$(\zeta^1(r), \ldots, \zeta^B(r)) \Rightarrow (\zeta^1, \ldots, \zeta^B) \tag{2.14}$$

in $D[0,\infty)^B$.

Proof. Fix $r \geq 1$ and $t \geq 0$. For all $1 \leq k \leq K$, we see from (2.11) that

$$\begin{aligned}
\zeta_t^k(r) &= \frac{V_{[rt]}^k(r) - U_{[rt]}}{\sqrt{r}} \\
&= \frac{V_{[rt]}^k(r) - v^k(r)[rt]}{\sqrt{r}} - \frac{U_{[rt]}^k(r) - u(r)[rt]}{\sqrt{r}} - \frac{[rt][u(r) - v^k(r)]}{\sqrt{r}} \\
&= \xi_t^k(r) - \xi_t^0(r) - \frac{[rt]}{r}[u(r) - v^k(r)]\sqrt{r}
\end{aligned}$$

From assumption **(Ib)** it is clear that as $r \uparrow \infty$,

$$\frac{[rt]}{r}[u(r) - v^k(r)]\sqrt{r} \to c_k t, \quad 1 \leq k \leq K$$

and we conclude to (2.14) by invoking Lemma 2.1 and the continuous mapping theorem [4, Theorem 5.1]. ∎

For $r = 1, 2 \ldots$, we define the stochastic processes $\eta^j(r) \equiv \{\eta_t^j(r), t \geq 0\}$ and $\mu^j(r) \equiv \{\mu_t^j(r), t \geq 0\}, 1 \leq j \leq B$, with sample paths in $D[0,\infty)$, by setting

$$\eta_t^j(r) = \frac{D_{[rt]}^j(r)}{\sqrt{r}}, \quad 1 \leq j \leq B, \quad t \geq 0 \tag{2.15}$$

and

$$\mu_t^j(r) = \frac{W_{[rt]}^j(r)}{\sqrt{r}}, \quad 1 \leq j \leq B, \quad t \geq 0. \tag{2.16}$$

The processes $\eta^j \equiv \{\mu_t^j, t \geq 0\}, 1 \leq j \leq B$, and $\mu^j \equiv \{\eta_t^j, t \geq 0\}, 1 \leq j \leq B$, are now defined by

$$\eta^j = g(\zeta^j - \max_{i \in p(j)} \eta^i) + \max_{i \in p(j)} \eta^i, \quad 1 \leq j \leq B \tag{2.17}$$

and

$$\mu^j = g(\zeta^j - \max_{i\in p(j)} \eta^i), \quad 1 \le j \le B. \tag{2.18}$$

In contrast with the situation for single stage fork–join queues, we note that the limiting processes (2.17)–(2.18) for acyclic fork–join networks are much more complicated.

Theorem 2.1 *As $r \uparrow \infty$,*

$$(\eta^1(r), \ldots, \eta^B(r)) \Rightarrow (\eta^1, \ldots, \eta^B) \tag{2.19}$$

in $D[0,\infty)^B$.

Before providing a proof for Theorem 2.1, we present the following two corollaries which identify the diffusion limit for the waiting times and the end-to-end delay of the system respectively.

Corollary 2.1: *As $r \uparrow \infty$,*

$$(\mu^1(r), \ldots, \mu^B(r)) \Rightarrow (\mu^1, \ldots, \mu^B) \tag{2.20}$$

in $D[0,\infty)^B$.

Proof. Note that for all $r = 1, 2 \ldots$,

$$W_n^j(r) = D_n^j(r) - \max_{i\in p(j)} \{D_n^i(r) + v_n^i(r)\}, \quad 1 \le j \le B \qquad\qquad n = 0, 1 \ldots$$

so that for all $r = 1, 2 \ldots$,

$$\mu_t^j(r) = \eta_t^j(r) - \max_{i\in p(j)} \{\eta_t^i(r) + \frac{v_{[rt]}^i(r)}{\sqrt{r}}\}, \quad 1 \le j \le B, \quad t \ge 0 \tag{2.21}$$

We obtain (2.20) from (2.19) and (2.21) by applying the continuous mapping theorem and the converging together theorem [4, Theorem 4.1].

∎

For $r = 1, 2 \ldots$, we introduce the stochastic processes $\kappa(r) \equiv \{\kappa_t(r), t \ge 0\}$ with sample paths in $D[0,\infty)$ by

$$\kappa_t(r) = \frac{T_{[rt]}(r)}{\sqrt{r}}, \quad t \ge 0. \tag{2.22}$$

Corollary 2.2 *As* $r \uparrow \infty$,

$$\kappa(r) \Rightarrow \max_{i \in p(B+1)} \eta^i \tag{2.23}$$

in $D[0, \infty)$.

Proof. Using the fact that for all $r = 1, 2 \dots$

$$\kappa_t(r) = \max_{i \in p(B+1)} \{\eta_t^i(r) + \frac{v_{[rt]}^i}{\sqrt{r}}\}, \quad t \geq 0 \tag{2.24}$$

we obtain (2.23) from (2.19) and (2.24) by applying the continuous mapping theorem and the converging together theorem. ∎

We now proceed with the proof for Theorem 2.1. For $1 \leq i \leq B$, we define the level $l(i)$ of queue i by

$$l(i) = \max_{j \in p(i)} l(j) + 1 \tag{2.15}$$

where by definition $l(i) = 1$ if $p(i) = \emptyset$. The level N of the graph is defined by $N := \max_{i \in V} l(i)$.

We denote the set of queues on level l as $q(l), 1 \leq l \leq N$ and assume that the cardinality of $q(l)$ is B_l. The queues are numbered in such a way that

$$\begin{aligned} q(1) &= \{1, \dots, B_1\} \\ q(2) &= \{B_1 + 1, \dots, B_1 + B_2\}, \\ &\vdots \\ q(N) &= \{B_1 + \dots + B_{N-1} + 1, \dots, B\} \end{aligned} \tag{2.26}$$

Note that the sets $q(1)$ and $q(N)$ have already been defined earlier as $s(0)$ and $p(B+1)$ respectively, with $B' = B_1$ and $B'' = B_1 + \dots + B_{N-1} + 1$

Proof. Our proof proceeds by induction on the levels of the acyclic graph which underlies the queueing network. First consider the queues belonging to the set $q(1)$, i.e.,queues j

such that $l(j) = 1$. Recall that for these queues $p(j) = \emptyset$, so that for $r = 1, 2 \ldots$ we have that

$$\begin{aligned} D^j_{n+1}(r) &= \max\{0, D^j_n(r) + v^j_n(r) - u_{n+1}(r)\} \\ &= S^j_{n+1}(r) - \min_{0 \le k \le n+1} S^j_k(r) \end{aligned} \qquad n = 0, 1 \ldots$$

so that

$$\eta^j_t(r) = g(\zeta^j(r))_t, \quad t \ge 0, \quad j = 1, \ldots, B_1, \tag{2.27}$$

upon taking note of (2.13) and (2.15). From (2.27) and (2.14) it follows that

$$(\eta^1(r), \ldots, \eta^{B_1}(r), \zeta^1(r), \ldots, \zeta^B(r)) \Rightarrow (\eta^1, \ldots, \eta^{B_1}, \zeta^1, \ldots, \zeta^B) \tag{2.28}$$

as $r \uparrow \infty$, so that (2.21) is verified for the queues belonging to the set $q(1)$.

As the induction hypothesis, assume that

$$(\eta^1(r), \ldots, \eta^{B_1+\ldots+B_l}(r), \zeta^1(r), \ldots, \zeta^B(r)) \Rightarrow (\eta^1, \ldots, \eta^{B_1+\ldots+B_l}, \zeta^1, \ldots, \zeta^B) \tag{2.29}$$

as $r \uparrow \infty$, which implies that (2.21) holds for the queues belonging to the first l levels. Using (2.29) we shall prove that (2.21) holds for queues belonging to the first $l + 1$ levels, thus completing the induction step.

Consider queue j such that $l(j) = l + 1$. Expanding the recursion in Lemma 2.1 for $r = 1, 2 \ldots$, $n = 0, 1 \ldots$ and $j = B_l + 1, \ldots, B_{l+1}$, we obtain

$$\begin{aligned} D^j_{n+1}(r) &= \max_{i \in p(j)} \{D^i_{n+1}(r) + v^i_{n+1}(r)\} \\ &\quad + \max\{0, D^j_n(r) - \max_{i \in p(j)} \{D^i_{n+1}(r) + v^i_{n+1}(r)\} + v^j_n(r) - u_{n+1}(r)\}, \\ &= \max_{i \in p(j)} \{D^i_{n+1}(r) + v^i_{n+1}(r)\} + S^j_{n+1}(r) - \max_{i \in p(j)} \{D^i_{n+1}(r) + v^i_{n+1}(r)\} \\ &\quad - \min_{0 \le k \le n+1} \{S^j_k(r) - \max_{i \in p(j)} \{D^i_k(r) + v^i_k(r)\}\} \end{aligned} \tag{2.30}$$

Note that by (2.30), we have for $j = B_l + 1, \ldots, B_{l+1}$ and $t \ge 0$,

$$\eta^j_t(r)$$

$$= \max_{i \in p(j)} \{\eta_t^i(r) + \frac{v^i_{[rt]}}{\sqrt{r}}\} + g\left(\zeta^j(r) - \max_{i \in p(j)}\{\eta^i(r) + \frac{v^i_{[r.]}}{\sqrt{r}}\}\right)_t \tag{2.31}$$

From (2.29), (2.31), the continuous mapping theorem and the converging together theorem, we conclude that as $r \uparrow \infty$,

$$(\eta^1(r), \ldots, \eta^{B_1+\ldots+B_{l+1}}(r), \zeta^1(r), \ldots, \zeta^B(r)) \Rightarrow (\eta^1, \ldots, \eta^{B_1+\ldots+B_{l+1}}, \zeta^1, \ldots, \zeta^B) \tag{2.32}$$

as $r \uparrow \infty$, which completes the induction step.

∎

3. An heuristic formula for the heavy traffic limit

In the last section we obtained a heavy traffic diffusion limits for the delay processes in an acyclic fork–join network. In order to obtain heavy traffic approximations, we now have to solve for the stationary distribution of this diffusion [23]. This seems to be a very difficult task due to the complicated nature of the diffusion. In another paper [24] we obtained approximations for the average response time for the special case of parallel fork–join queues. These approximations were based on the following formula for the heavy traffic limit that was proposed in [24]. Consider a K–dimensional parallel fork–join queue subject to an arrival process with variance σ_0^2 and rate λ and a service process with variance σ^2 and rate μ. Then the heavy traffic limit for the average response time $\overline{T}_K(\lambda)$ is given by

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}_K(\lambda)$$

$$= \left[H_K + (4V_K - 3H_K - 1)\beta + 2(1 + H_K - 2V_K)\beta^2\right] \frac{\sigma^2 + \sigma_0^2}{2}\mu^2, \quad 0 \le \beta \le 1. \qquad K = 2, 3 \ldots \tag{3.1}$$

where

$$\beta = \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2}, \quad 0 \le \beta \le 1. \tag{3.2}$$

Equation (3.1) was validated by extensive simulations in [24] and was shown to give extremely good results. It was obtained by using the following line of heuristic reasoning:

Note that the parameter β always lies between 0 and 1. The heavy traffic limit is easy to obtain in the cases $\beta = 0$ (deterministic arrivals) and $\beta = 1$ (deterministic services). The heavy traffic limit for the case $\beta = \frac{1}{2}$ is obtained by postulating the following equality between the heavy traffic limit and light traffic derivative, for the case when the arrivals are Poisson and services are exponential,

$$\mu^2 \overline{T}'(0) = \lim_{\lambda \uparrow \mu} (\mu - \lambda) \overline{T}(\lambda). \tag{3.3}$$

Having obtained the heavy traffic limit for $\beta = 0, \frac{1}{2}$ and 1, we carried out a quadratic interpolation in β, for $\beta \in [0, 1]$ to finally arrive at (3.1).

We plan to carry out a similar program in order to obtain a formula for the heavy traffic limit for homogeneous acyclic fork–join networks. Hence we start with the hypothesis that the heavy traffic limit for homogeneous acyclic fork–join networks is given by

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda) \overline{T}(\lambda) = \left[A + B\beta + C\beta^2\right] \frac{\sigma^2 + \sigma_0^2}{2} \mu^2, \quad 0 \leq \beta \leq 1 \tag{3.4}$$

where the values of the constants A, B and C depend upon the topology of the network. However, unlike in the case of parallel fork–join queues, exact heavy traffic limits are unavailable even for the cases $\beta = 0$ or 1. Hence we will have to rely exclusively on (3.3) to obtain the heavy traffic limits with the help of the light traffic derivative.

The light traffic derivative can be obtained by simulating the system near $\lambda = 0$. As pointed out earlier, in [24] we assumed that (3.3) was valid only for Poisson arrivals and exponential services. However note that for the case of a single server queue, it holds for Poisson arrivals and *general* service distributions, since

$$\mu^2 \overline{T}'(0) = \lim_{\lambda \uparrow \mu} (\mu - \lambda) \overline{T}(\lambda) = \frac{1 + \mu^2 \sigma^2}{2}. \tag{3.5}$$

Moreover, in [24] we made the observation that (3.3) was satisfied for parallel fork–join queues subject to Poisson arrivals, whose service times had coefficient of variation less than one. When the coefficient of variation exceeded one (as in the case of hyper–exponential services), then (3.3) was no longer true. However, for the particular fork–join network that we chose to analyze in this paper, it was observed that (3.3) continues to hold even when coefficient of variation of the service time exceeds one. The following conclusions may be drawn from this:

(a): Equation (3.3) is satisfied for acyclic fork–join networks subject to Poisson arrivals, whose service times have coefficient of variation less than one.

(b): Equation (3.3) may not satisfied in case the coefficient of variation of the service times exceeds one (with the arrivals still being Poisson). However, this varies according to the topology of the network, and there are certain networks for which (3.3) holds even for general service times.

We shall illustrate these ideas by considering the example of the homogeneous fork–join network depicted in Fig 2. Our plan is to obtain the heavy traffic limits at $\beta = 0, \frac{1}{2}$ and 1, and then combine them in the manner discussed above, in order to obtain a heavy traffic formula for $\beta \in [0,1]$.

(1): The case $\beta = \frac{1}{2}$. Assume that the arrivals into the system constitute a Poisson stream with rate λ, while the service time distribution at each queue is exponential with rate μ, so that $\beta = \frac{1}{2}$. Based on our experience with the parallel fork–join queue, we would expect that the light traffic limits for the average response time $\overline{T}(\lambda)$ are given by

$$\overline{T}(0) = \frac{C}{\mu} \tag{3.6}$$

$$\overline{T}'(0) = \frac{D}{\mu^2} \tag{3.7}$$

where C and D are constants. The heavy traffic limit can be obtained with the help of (3.3), so that

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}(\lambda) = D \tag{3.8}$$

The only task left is to identify the value of the constants C and D. Even though it is possible to calculate them using the light traffic theory, the calculations are quite tedious for most networks. A more viable method would be automate the calculations using some symbolic computation tool such as MACSYMA. However, this is a subject for future research and here we only give a simulation based technique for obtaining C and D. The main idea in this technique is to simulate the system with $\mu = 1$, and use the simulation output for obtaining C and D. For e.g., for the network in Fig 2, for the case when the services are exponential with rate $\mu = 1$, we found that the average response time $\overline{T}(\lambda)$ satisfies

$$\overline{T}(0.0005) = 5.05 \text{ and } \overline{T}(0.01) = 5.10$$

so that

$$C = 5.05 \text{ and } D = 4.35$$

Hence it follows that

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}(\lambda) = 4.35 \tag{3.9}$$

If we substitute $\beta = \frac{1}{2}$ and $\sigma^2 = \sigma_0^2 = \frac{1}{\mu^2}$ into (3.5), we obtain

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}(\lambda) = A + 0.5B + 0.25C \tag{3.10}$$

Hence, by (3.9) and (3.10) it follows that

$$A + 0.5B + 0.25C = 4.35. \tag{3.11}$$

We now validate (3.9) by comparing its predictions with simulation data. Combining (3.6), (3.7) and (3.8) we obtain the following approximation for the average response time for the case of Poisson arrivals and exponential services.

$$\hat{T}(\lambda) = \left[5.06 - 0.71\frac{\lambda}{\mu}\right] \frac{1}{\mu - \lambda} \tag{3.12}$$

This approximation is compared with simulations in Section 3.1 and as the reader may note, the agreement is quite good.

(2): The case $\beta = 1$. Consider a fork–join queue with Poisson arrivals and deterministic services (so that $\beta = 1$). Carrying out the procedure exactly as for $\beta = \frac{1}{2}$, it can be shown that

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}(\lambda) = 0.53 \tag{3.13}$$

If we substitute $\beta = 1, \sigma^2 = 0$ and $\sigma_0^2 = \frac{1}{\mu^2}$ into (3.5), we obtain

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}(\lambda) = \frac{A + B + C}{2} \tag{3.14}$$

Combining (3.13) and (3.14), we conclude that

$$A + B + C = 1.26 \tag{3.15}$$

Equation (3.13) can be validated by comparing the interpolation approximation,

$$\hat{T}(\lambda) = \left[4 - 3.47\frac{\lambda}{\mu}\right] \frac{1}{\mu - \lambda} \tag{3.14}$$

for the case of Poisson arrivals and deterministic services with simulation. This approximation also compare extremely well with simulation results as shown in Section 3.1.

(3): The case $\beta = 0$. Note that $\beta = 0$ corresponds to deterministic arrivals, so that (3.3) is no longer valid. However if we retain the assumption of Poisson arrivals, but choose a service time with large variance, then $\beta \approx 0$. To that end, assume that the arrivals are Poisson and that the service times possess the following hyper–exponential density function

$$f_S(x) = p_1 \mu_1 e^{-\mu_1 x} + p_1 \mu_2 e^{-\mu_2 x}, \quad x \geq 0. \tag{3.17}$$

Assume that the condition

$$\frac{p_1}{\mu_1} = \frac{p_2}{\mu_2} = \frac{1}{2} \tag{3.18}$$

is satisfied. Then note that

$$ES = 1 \text{ and } ES^2 = \frac{1}{\mu_1} + \frac{1}{\mu_2} \tag{3.19}$$

so that the square of the coefficient of variation, c_S^2 is given by

$$c_S^2 = \frac{1}{\mu_1} + \frac{1}{\mu_2} - 1 = \frac{2}{\mu_1 \mu_2} - 1 \tag{3.20}$$

We shall choose $p_1 = 0.005, p_2 = 0.995, \mu_1 = 0.01$ and $\mu_2 = 1.99$, so that $\sigma^2 = 100.51$ and $\beta = 0.0098$. Once again, carrying out the calculations as in **(1)**, we obtain

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}(\lambda) = 347.05. \tag{3.21}$$

If we substitute $\beta = 0, \sigma_0^2 = 1$ and $\mu = 1$ into (3.5), we obtain

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}(\lambda) = \frac{A(\sigma_0^2 + \sigma^2)}{2} \tag{3.22}$$

Combining (3.21) and (3.22), we conclude that

$$A = 6.84 \tag{3.23}$$

Equation (3.21) can be validated by comparing the interpolation approximation,

$$\hat{T}(\lambda) = \left[6.14 + 340.91\frac{\lambda}{\mu}\right]\frac{1}{\mu - \lambda} \tag{3.24}$$

for the case of Poisson arrivals and hyper–exponential services with simulation. This approximation also compare extremely well with simulation results as shown in Section 3.1.

Combining (3.11), (3.15) and (3.23), we come to the conclusion that

$$A = 6.84, \quad B = -4.38 \quad \text{and} \quad C = -1.2$$

so that (3.5) takes the form

$$\lim_{\lambda \uparrow \mu} (\mu - \lambda)\overline{T}(\lambda) = \left[6.84 - 4.38\beta - 1.2\beta^2\right]\frac{\sigma^2 + \sigma_0^2}{2}\mu^2, \quad 0 \leq \beta \leq 1 \tag{3.25}$$

for the acyclic fork–join network under consideration.

In the remainder of this paper we validate (3.25) by using to obtain approximations for the following two cases:

(i): Consider the fork–join network subject to Poisson arrivals and Erlang–2 service distributions. For this case we have $\beta = \frac{2}{3}, \sigma^2 = \frac{1}{2\mu^2}$ and $\sigma_0^2 = \frac{1}{\mu^2}$. Substituting these values into (3.25), we obtain

$$\lim_{\lambda \uparrow \mu}(\mu - \lambda)\overline{T}(\lambda) = 2.54 \tag{3.26}$$

Combining this heavy traffic limit with the light traffic limit $\overline{T}(0)$ (which is obtained by simulating the system), we obtain the approximation

$$\hat{T}(\lambda) = \left[4.76 - 2.22\frac{\lambda}{\mu}\right]\frac{1}{\mu - \lambda} \tag{3.27}$$

This approximation is compared with simulation in Section 3.1 and good agreement is observed.

(ii): Consider the fork–join network subject to Poisson arrivals and hyper–exponential service distribution with $p_1 = 0.05, p_2 = 0.95, \mu_1 = 0.1$ and $\mu_2 = 1.9$. For this case we have $\beta = 0.095, \sigma^2 = 9.526$ and $\sigma_0^2 = 1$. Substituting these values into (3.25), we obtain

$$\lim_{\lambda \uparrow \mu}(\mu - \lambda)\overline{T}(\lambda) = 33.75 \tag{3.28}$$

Combining this heavy traffic limit with the light traffic limit $\overline{T}(0)$ (which is obtained by simulating the system), we obtain the approximation

$$\hat{T}(\lambda) = \left[5.8 + 27.95\frac{\lambda}{\mu}\right]\frac{1}{\mu - \lambda} \tag{3.29}$$

This approximation is compared with simulation in Section 3.1 and good agreement is observed.

3.1 Simulation results

Approximation (3.12) compared with simulation results (the case of Poisson arrivals and exponential services).

λ	$\overline{T}(\lambda)$	$\hat{T}(\lambda)$	% Error
0.1	5.56 ± 0.018	5.54	0.28
0.2	6.19 ± 0.026	6.141	0.73
0.3	7.02 ± 0.039	6.926	1.38
0.4	8.11 ± 0.058	7.96	1.91
0.5	9.63 ± 0.090	9.41	2.31
0.6	11.92 ± 0.152	11.59	2.87
0.7	15.73 ± 0.285	15.21	3.33
0.8	23.71 ± 0.215	22.46	5.29
0.9	46.93 ± 0.76	44.21	5.80

Approximation (3.16) compared with simulation results (the case of Poisson arrivals and deterministic services).

λ	$\overline{T}(\lambda)$	$\hat{T}(\lambda)$	% Error
0.1	5.06 ± 0.010	5.04	0.28
0.2	5.44 ± 0.017	5.40	0.66
0.3	5.93 ± 0.024	5.87	1.01
0.4	6.57 ± 0.034	6.48	1.35
0.5	7.47 ± 0.054	7.34	1.74
0.6	8.84 ± 0.093	8.63	2.37
0.7	11.19 ± 0.189	10.78	3.66
0.8	15.89 ± 0.123	15.09	5.03
0.9	30.13 ± 0.496	28.00	7.06

Approximation (3.24) compared with simulation results (the case of Poisson arrivals and hyper–exponential services).

λ	$\overline{T}(\lambda)$	$\hat{T}(\lambda)$	% Error
0.1	45.82 ± 0.98	44.70	2.44
0.2	99.45 ± 2.44	92.90	6.58
0.3	170.58 ± 4.62	154.87	9.21
0.4	266.26 ± 7.48	237.51	10.79
0.5	398.29 ± 11.28	353.19	11.33
0.6	593.77 ± 16.04	526.71	11.29
0.7	903.09 ± 22.54	815.92	9.65
0.8	1480.83 ± 32.96	1394.34	5.84
0.9	3162.33 ± 57.88	3129.59	1.03

Approximation (3.27) compared with simulation results (the case of Poisson arrivals and Erlang–2 services).

λ	$\overline{T}(\lambda)$	$\hat{T}(\lambda)$	% Error
0.1	5.06 ± 0.010	5.04	0.39
0.2	5.44 ± 0.017	5.39	0.92
0.3	5.93 ± 0.024	5.85	1.34
0.4	6.57 ± 0.034	6.45	1.83
0.5	7.47 ± 0.054	7.30	2.27
0.6	8.84 ± 0.093	8.57	3.05
0.7	11.19 ± 0.189	10.69	4.47
0.8	15.89 ± 0.123	14.92	6.10
0.9	30.13 ± 0.496	27.62	8.33

Approximation (3.29) compared with simulation results (the case of Poisson arrivals and hyper–exponential services).

λ	$\overline{T}(\lambda)$	$\hat{T}(\lambda)$	% Error
0.1	9.90 ± 0.044	9.55	3.53
0.2	15.28 ± 0.088	14.24	6.80
0.3	22.39 ± 0.156	20.26	9.51
0.4	31.90 ± 0.155	28.30	11.28
0.5	45.16 ± 11.28	39.55	12.42
0.6	64.67 ± 0.428	56.42	12.75
0.7	96.42 ± 0.733	84.55	12.31
0.8	157.76 ± 2.85	140.80	10.75
0.9	334.76 ± 3.16	309.55	7.53

REFERENCES

[1] F. Baccelli, A.M. Makowski and A. Shwartz, "Fork-Join queue and related systems with synchronization constraints: Stochastic ordering, approximations and computable bounds," Electrical Engineering Technical Report, No. TR-87-01 University of Maryland, College Park (1987).

[2] F. Baccelli, W.A. Massey and D. Towsley, "Acyclic fork-join queueing networks," *INRIA Rapport de Recherche*, No. 688 (1987).

[3] F. Baccelli and A.M. Makowski, "Queueing systems with synchronization constraints," *Proceeding of the IEEE*, (1989).

[4] P. Billingsley, *"Convergence of probability measures,"* J. Wiley and Sons, New York,

(1968).

[5] E.G. Coffman Jr. and M.I. Reiman,"Diffusion approximations for computer/ communication systems," *Mathematical Computer Performance and Reliability*, G. Iazeolla, P.J. Courtois and A. Hordijk (editors), North Holland (1984).

[6] J.M. Harrison, *Brownian motion and stochastic flow systems*, J. Wiley and Sons, New York (1985).

[7] J.M. Harrison and R.J. Williams,"Brownian models of open queueing networks with homogeneous customer population," *Stochastics*, Vol. 22, pp. 77-115 (1987).

[8] D.L. Iglehart and W. Whitt, "Multiple channel queues in heavy traffic. I," *Adv. Appl. Prob.*, No. 2, pp. 150-177 (1970).

[9] D.L. Iglehart and W. Whitt, "Multiple channel queues in heavy traffic. II: Sequences, networks and batches," *Adv. Appl. Prob.*, No. 2, pp.355-369 (1970).

[10] D.L. Iglehart, "Weak convergence in queueing theory," *Adv. Appl. Prob.*, Vol. 5, pp. 570-594 (1973).

[11] E. Kyprianou, "The virtual waiting time of the $GI/G/1$ queue in heavy traffic," *Adv. Appl. Prob.*, No. 3, pp. 249-268 (1971).

[12] A.M. Law and W.D. Kelton, *Simulation modeling and analysis*, Mc-Graw Hill, New York (1982).

[13] A.J. Lemoine, "Networks of queues-A survey of weak convergence results," *Manag. Sci.*, Vol. 24, No. 11, pp. 1175-1193 (1978).

[14] T. Lindvall,"Weak convergence of probability measures and random functions in the function space $D[0,\infty)$," *J. Appl. Prob.*, Vol. 10, pp. 109-121 (1973).

[15] K.R. Parthasarthy, *Probability measures on metric spaces*, Academic Press, New York (1967).

[16] Y. Prohorov,"Convergence of random processes and limit theorems of probability theory," *Theor. Probability Appl.*, Vol. 1, pp. 157-214 (1956).

[17] Y. Prohorov,"Transient phenomena in processes of mass service," *Litovsk. Mat. Sb.*, Vol. 3, pp. 199-205 (1963).

[18] M.I. Reiman, "Open queueing networks in heavy traffic," *Maths. of Oper. Res.*, Vol. 9, No. 3, pp. 441-458 (1984).

[19] M.I. Reiman and B. Simon, "An interpolation approximation for queueing systems with Poisson input," *Oper. Res.*, Vol. 36, No. 3, pp. 454-469 (1988).

[20] M.I. Reiman and B. Simon, "Light traffic limits of sojourn time distributions in Markovian queuing networks," *Commun. Statist.-Stochastic Models*, Vol. 4, No. 2, pp. 191-233 (1988).

[21] M.I. Reiman and B. Simon, "Open queueing systems in light traffic," *Maths. of Oper. Res.*, Vol. 14, No. 1, pp. 26-59 (1989).

[22] S. Varma, "Heavy and light traffic approximations for queues with synchronization constraints," PhD Thesis, University of Maryland, (1990).

[23] S. Varma and A.M. Makowski, "Heavy traffic limits for fork–join queues," In preparation, (1990).

[24] S. Varma and A.M. Makowski, "Interpolation approximations for fork–join queues," In preparation, (1990).

[25] A.M. Makowski, T.K. Philips and S. Varma,"Comparison of scheduling strategies in multiprocessor systems," In preparation (1990).

[26] S. Varma and A.M. Makowski, "Approximations for synchronized networks," In preparation, (1990).

[27] W.Whitt,"Weak convergence theorems for queues in heavy traffic," Ph.D. Thesis, Cornell University (1968).

[28] W. Whitt,"Weak convergence of probability measures on the function space $C[0,\infty)$." *Ann. Math. Stat.*, Vol. 41, No. 3, pp. 939-944 (1970).

[29] W. Whitt,"Heavy traffic limit theorems for queues: A survey," Lecture Notes in Economics and Mathematical Systems, No. 98, Springer-Verlag, Berlin, pp. 307-350 (1974).

[30] W. Whitt,"Some useful functions for functional limit theorems," *Maths. of Oper. Res*, Vol. 5, No. 1, pp. 67-85 (1980). No. 98, Springer-Verlag, Berlin, pp. 307-350 (1974).

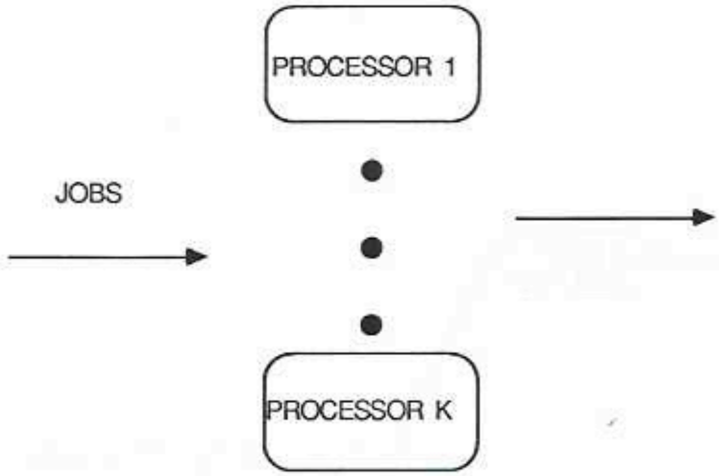

Fig. 1(a). The system of parallel processors

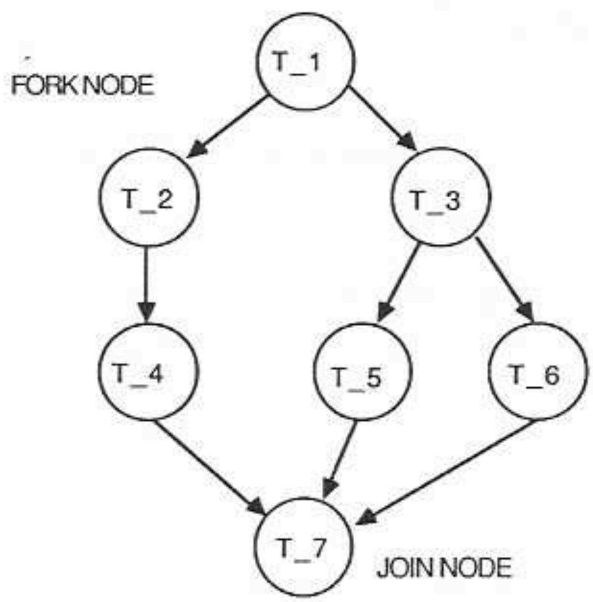

Fig. 1(b). A job model consisting of seven tasks

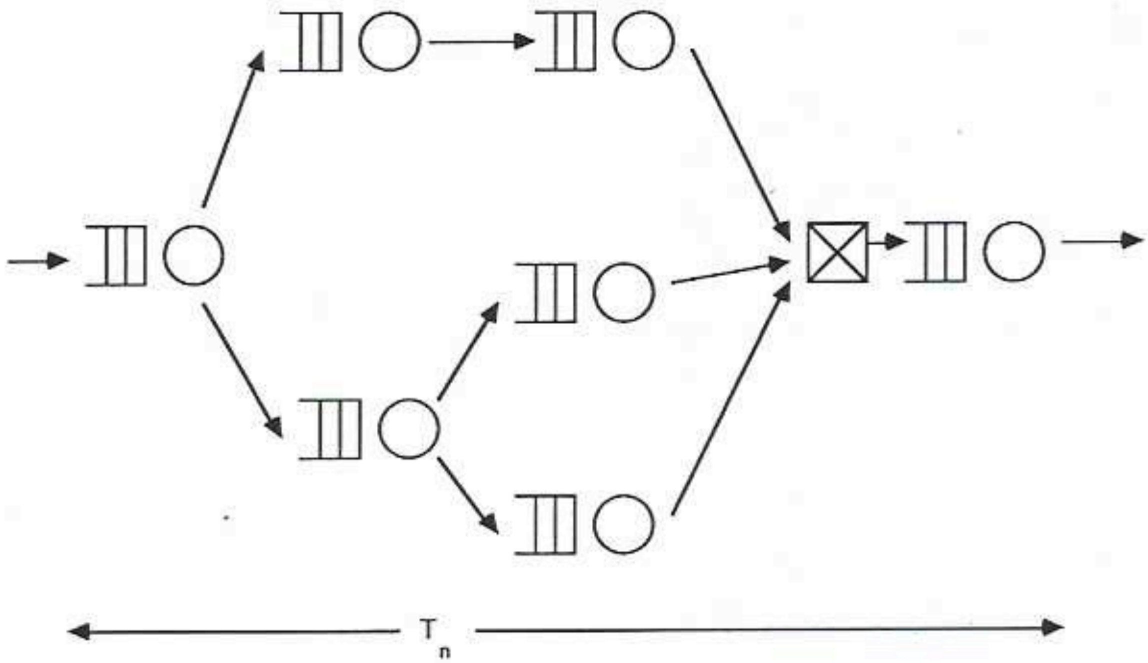

Fig. 2. An example of an acyclic fork-join network